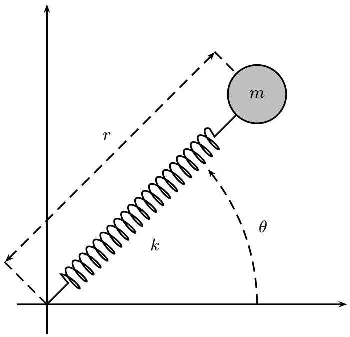
Symmetry and reduction of Lagrangian systems

Tom Mestdag

University of Antwerp

An example



The Lagrangian is $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}kr^2$.

The Euler-Lagrange equations are

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{r}}\right) - \dfrac{\partial L}{\partial r} = 0 \Rightarrow m\ddot{r} - mr\dot{\theta}^2 + kr = 0. \\ \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right) - \dfrac{\partial L}{\partial \theta} = 0 \Rightarrow mr^2\dot{\theta} = \mu. \end{cases}$$

(For a central force angular momentum is conserved)

Fix μ. The first equation becomes $m\ddot{r} - mr\left(\dfrac{\mu}{mr^2}\right)^2 + kr = 0$.

$\rightsquigarrow$ This is again an Euler-Lagrange equation, for the Lagrangian

$$\begin{aligned} L_\mu(r, \dot{r}) &= \frac{1}{2}m\dot{r}^2 - \frac{1}{2}kr^2 - \frac{1}{2}\frac{\mu^2}{mr^2} \\ &= \left(L - \frac{\partial L}{\partial \dot{\theta}}\dot{\theta}\right)_{\dot{\theta}=\mu/(mr^2)}. \end{aligned}$$

$\rightsquigarrow$ Solve it for r, and find θ from $mr^2\dot{\theta} = \mu$.

Example: Poisson reduction

Definition. A *Poisson bracket* on M is a bilinear map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

satisfying:
- skew-symmetry: $\{F, G\} = -\{G, F\}$.
- Jacobi: $\{F, \{G, H\}\} + \text{ cycl} = 0$.
- Leibniz: $\{FG, H\} = F\{G, H\} + G\{F, H\}$.

For a function H, $X_H : F \mapsto \{F, H\}$ is its *Hamiltonian vector field*.

A smooth map $\Phi : M \to B$ between Poisson manifolds is *Poisson* if

$$\{f, g\}_B \circ \Phi = \{f \circ \Phi, g \circ \Phi\}_M.$$

Let $M \to B = M/G$ be a principal fibre bundle (the action Φ is free and proper).

The action Φ is *Poisson* if each Φ_g is a Poisson map.

Theorem. *Given a Poisson action. Then also $B = M/G$ is Poisson with*

$$\{f, g\}_B \circ \pi = \{f \circ \pi, g \circ \pi\}_M, \quad \forall f, g, \in \mathcal{F}(B).$$

When H is a G-invariant Hamiltonian on M, with $H = h \circ \pi$.

Then, $Y = X_H \in \mathcal{X}(M)$ is G-invariant and its reduced vector field is $\bar{Y} = X_h \in \mathcal{X}(B)$.

In this case, symmetry reduction is structure preserving!

Proof. The verification of the bracket is immediate.
By definition, the flow ϕ_t^H of X_H on M satisfies

$$\frac{d}{dt}(F \circ \phi_t^H(m)) = (X_H(F) \circ \phi_t^H)(m).$$

With $X_H(F) = \{F, H\}_M$ and $H = h \circ \pi$, we get, for $F = f \circ \pi$,

$$\frac{d}{dt}(f \circ \pi \circ \phi_t^H(m)) = \{f, h\}_B \circ \pi \circ \phi_t^H(m).$$

Analogously, the flow ϕ_t^h of X_h satisfies

$$\frac{d}{dt}(f \circ \phi_t^h([m])) = \{f, h\}_B \circ \phi_t^h([m]).$$

We conclude that $\overline{\phi}_t \circ \pi = \pi \circ \phi_t$, and thus

$$\pi \circ \phi_t \circ \Phi_g = \overline{\phi}_t \circ \pi \circ \Phi_g = \pi \circ \phi_t,$$

Due to freeness: $\phi_t \circ \Phi_g = \Phi_g \circ \phi_t$. Therefore X_H is G-invariant, and it reduces to X_h.

Coordinate expressions

- Let $(x^\alpha) = (x^i, x^a)$ be coordinates on M,

  (x^i) coordinates on $B = M/G$.

- Let $\{E_a\}$ be a basis for $\mathfrak{g}$, then $\{(E_a)_M = \tilde{E}_a\}$ is a frame for $\mathcal{V}$.

  $\rightsquigarrow \tilde{E}_a = K_a^b(x)\dfrac{\partial}{\partial x^b}$, with $\det(K_a^b) \neq 0$.

  $\rightsquigarrow [\tilde{E}_a, \tilde{E}_b] = -C_{ab}^c \tilde{E}_c$.

- Take a principal connection $\mathcal{A}$. Let

$$X_i = \left(\frac{\partial}{\partial x^i}\right)^H = \frac{\partial}{\partial x^i} - \gamma_i^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial x^i} - \Lambda_i^c \tilde{E}_c.$$

  $\rightsquigarrow [\tilde{E}_a, X_i] = 0 \quad$ and $\quad [X_i, X_j] = R_{ij}^a \tilde{E}_a$.

• For $Y = Y^i X_i + Z^b \tilde{E}_b = Y = Y^i \frac{\partial}{\partial x^i} + \left(Z^b - Y^i \Lambda_i^b\right) K_b^a \frac{\partial}{\partial x^a}$,

we get $\mathcal{A}(Y) = Z^b E_b$.

• When Y is invariant then $[\tilde{E}_a, Y] = 0$, or

$$\tilde{E}_a(Y^i) = 0 = \frac{\partial Y^i}{\partial x^a}, \qquad \tilde{E}_a(Z^b) - Z^c C_{ac}^b = 0.$$

With this, the equations for the integral curves become

$$\begin{cases} \dot{x}^i = Y^i, \\ \dot{x}^a = (Z^b - \Lambda_i^b \dot{x}^i) K_b^a. \end{cases}$$

• The reduced vector field is $\bar{Y} = Y^i \frac{\partial}{\partial x^i}$, with equations for integral curves

$$\dot{x}^i = Y^i.$$

The rest of the course is mainly based on

- T. Mestdag and M. Crampin, Invariant Lagrangians mechanical connections and the Lagrange-Poincare equations, *Journal of Physics A: Mathematical and Theoretical* 41 (2008), 344015.
- M. Crampin and T. Mestdag, Routh's procedure for non-Abelian symmetry groups, *Journal of Mathematical Physics* 49 (2008), 032901.
- M. Crampin and T. Mestdag, Relative equilibria of Lagrangian systems with symmetry, *Journal of Geometry and Physics* 58 (2008) 874–887.

Coordinates for $\pi : M \to B = M/G$.

- $(x^\alpha) = (x^i, x^a)$ are coordinates on M,
- $(x^\alpha, \dot{x}^\alpha)$ are coordinates on TM,
- (x^i) are coordinates on $B = M/G$,
- $(x^i, \dot{x}^i)$ are coordinates on $T(M/G)$,
- Coordinates on $(TM)/G$? We will use "quasi-velocities".

An invariant local frame of vector fields on M

We will use a G-**invariant** frame $\{Z_\alpha\}=\{\hat{E}_a, X_i\}$ of VFs on M, where

1. $\hat{E}_a$, $a = 1 \ldots \dim(G)$ are tangent to the fibres of $M \to M/G$;
2. X_i, $i = 1 \ldots \dim(M/G)$ are transverse to the fibres.

How are they defined?

1. Let $\{E_a\}$ be a basis of $\mathfrak{g}$ and $\tilde{E}_a$ the corresponding (not-invariant) infin. generators. Then $[\tilde{E}_a, \tilde{E}_b] = -C^c_{ab}\tilde{E}_c$.

$\rightsquigarrow$ A vector field $\hat{E}_a = A^b_a \tilde{E}_b$ is invariant if

$$0 = [\tilde{E}_a, \hat{E}_b] = \Big(\tilde{E}_a(A^c_b) - C^c_{ad}A^d_b\Big)\tilde{E}_c.$$

The integrability condition corresponds to the Jacobi identity.

$\rightsquigarrow$ There are local solutions, for which $A = (A^b_a)$ is non-singular, and for which A is the identity on some specified local section of π.

$\rightsquigarrow$ Let $U \subset M/G$ be an open set over which M is locally trivial and let (x^i) be coordinates on M/G. Then $\pi : U \times G \to U$, and $\psi^M_g(x, h) = (x, gh)$ and we may define

$$\hat{E}_a : (x, g) \mapsto (\widetilde{\mathrm{ad}_{g^{-1}} E_a})(x, g) = T\psi^M_g\big(\tilde{E}_a(x, e)\big).$$

2. Assume a principal connection on $M \to M/G$ given; take X_i to be the horizontal lift of $\partial/\partial x^i$ on M/G.

Conclusion: There is an invariant frame $\{X_i, \hat{E}_a\}$.

The Lie brackets of the members of the frame take the form

$$[\hat{E}_a, \hat{E}_b] = C^c_{ab}\hat{E}_c, \quad [X_i, X_j] = K^a_{ij}\hat{E}_a, \quad [X_i, \hat{E}_a] = \Upsilon^b_{ia}\hat{E}_b.$$

We may lift the frame $\{X_i, \hat{E}_a\}$ on M to the frame $\{X_i^{\mathrm{C}}, \hat{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}, \hat{E}_a^{\mathrm{V}}\}$ on TM.

We know that $[\tilde{E}_a, X_i] = 0$ and $[\tilde{E}_a, \hat{E}_b] = 0$, and therefore

$$\begin{aligned}
[\tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{C}}] &= [\tilde{E}_a, X_i]^{\mathrm{C}} = 0, \\
[\tilde{E}_a^{\mathrm{C}}, X_i^{\mathrm{V}}] &= [\tilde{E}_a, X_i]^{\mathrm{V}} = 0, \\
[\tilde{E}_a^{\mathrm{C}}, \hat{E}_b^{\mathrm{C}}] &= [\tilde{E}_a, \hat{E}_b]^{\mathrm{C}} = 0, \\
[\tilde{E}_a^{\mathrm{C}}, \hat{E}_b^{\mathrm{V}}] &= [\tilde{E}_a, \hat{E}_b]^{\mathrm{V}} = 0.
\end{aligned}$$

So: these are all invariant vector fields on TM, and therefore reduce to vector fields $\bar{X}_i^{\mathrm{C}}$, $\bar{X}_i^{\mathrm{V}}$, $\bar{E}_a^{\mathrm{C}}$, $\bar{E}_a^{\mathrm{V}}$ on TM/G.

How do they look like? We need coordinates on TM/G.

Corresponding quasi-velocities

Quasi-velocities are fibre coordinates (v^i, w^a) in T_mM w.r.t. $\{X_i, \hat{E}_a\}$:

$$v_m = v^i X_i(m) + w^a \hat{E}_a(m).$$

If we write $v_m = \dot{x}^i \frac{\partial}{\partial x^i}\mid_m + \dot{x}^a \frac{\partial}{\partial x^a}\mid_m$, then $v^i = \dot{x}^i$ and $w^a = \ldots$

Coordinates on TM: (x^i, x^a, v^i, w^a).

Idea. We will show that the coordinate functions x^i, v^i and w^a are invariant functions on TM, e.g. $\tilde{E}_b^C(w^a) = 0$.

As a consequence: (x^i, v^i, w^a) can be used as coordinates on TM/G.

Recall: For any vector field Z, function f and 1-form θ on M,

$$Z^{\mathrm{C}}(f) = Z(f), \quad Z^{\mathrm{C}}(\vec{\theta}) = \overrightarrow{\mathcal{L}_Z\theta}, \quad Z^{\mathrm{V}}(f) = 0, \quad Z^{\mathrm{V}}(\vec{\theta}) = \theta(Z).$$

Let $\{\varpi^a, \vartheta^i\}$ be the dual 1-form basis of $\{\hat{E}_a, X_i\}$. Then $\vec{\varpi}^a = w^a$ and $\vec{\vartheta}^i = v^i$.

We have, for example,

$$\begin{aligned}
(\mathcal{L}_{\tilde{E}_b}\varpi^a)(\hat{E}_c) &= \tilde{E}_b(\delta^a_c) - \varpi^a([\tilde{E}_b, \hat{E}_c]) = 0 \\
(\mathcal{L}_{\tilde{E}_b}\varpi^a)(X_j) &= -\varpi^a([\tilde{E}_a, X_j]) = 0,
\end{aligned}$$

so $\tilde{E}_b^{\mathrm{C}}(w^a) = 0$.

Likewise $\tilde{E}_b^{\mathrm{C}}(x^i) = 0$ and $\tilde{E}_b^{\mathrm{C}}(v^i) = 0$.

Conclusion: we may use (x^i, v^i, w^a) as coordinates on TM/G.

Invariant vector fields on TM

An invariant vector field W on TM projects onto a vector field $\bar{W}$ on TM/G, so that

$$T\pi^{TM} \circ W = \bar{W} \circ \pi^{TM}.$$

Furthermore, if F is an invariant function on TM, the $F = \bar{F} \circ \pi$ for some function $\bar{F}$ on TM/G. Then

$$W(F) = W(\bar{F} \circ \pi^{TM}) = \bar{W}(\bar{F}) \circ \pi^{TM}.$$

Idea: Since (x^i, v^i, w^a) are coordinates on TM/G, the action of W on these completely determines $\bar{W}$.

We do this for $W = X_i^{\mathrm{C}}, X_i^{\mathrm{V}}, \hat{E}_a^{\mathrm{V}}$ and $F = x^i, v^i, w^a$.

The reduced frame on TM

For any vector field Z, function f and 1-form θ on M,

$$Z^{\mathrm{C}}(f) = Z(f), \quad Z^{\mathrm{C}}(\vec{\theta}) = \overrightarrow{\mathcal{L}_Z\theta}, \quad Z^{\mathrm{V}}(f) = 0, \quad Z^{\mathrm{V}}(\vec{\theta}) = \theta(Z).$$

Let $\{\varpi^a, \vartheta^i\}$ be the dual 1-form basis of $\{\hat{E}_a, X_i\}$. Then $\vec{\varpi}^a = w^a$ and $\vec{\vartheta}^i = v^i$.

We have, for example,

$$\begin{aligned} (\mathcal{L}_{X_i}\varpi^a)(\hat{E}_b) &= X_i(\delta^a_b) - \varpi^a([X_i, \hat{E}_b]) = -\Upsilon^a_{ib} \\ (\mathcal{L}_{X_i}\varpi^a)(X_j) &= -\varpi^a([X_i, X_j]) = -K^a_{ij}, \end{aligned}$$

so $X_i^{\mathrm{C}}(w^a) = -K^a_{ij}v^j - \Upsilon^a_{ib}w^b$.

The relevant derivatives are

$$
\begin{aligned}
&X_i^{\mathrm{C}}(x^j) = \delta_i^j, && X_i^{\mathrm{C}}(v^j) = 0, && X_i^{\mathrm{C}}(w^a) = -K_{ij}^a v^j - \Upsilon_{ib}^a w^b,\\
&X_i^{\mathrm{V}}(x^j) = 0, && X_i^{\mathrm{V}}(v^j) = \delta_i^j, && X_i^{\mathrm{V}}(w^a) = 0,\\
&\hat{E}_a^{\mathrm{C}}(x^j) = 0, && \hat{E}_a^{\mathrm{C}}(v^i) = 0, && \hat{E}_a^{\mathrm{C}}(w^b) = \Upsilon_{ia}^b v^i + C_{ac}^b w^c,\\
&\hat{E}_a^{\mathrm{V}}(x^j) = 0, && \hat{E}_a^{\mathrm{V}}(v^i) = 0, && \hat{E}_a^{\mathrm{V}}(w^b) = \delta_a^b,
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T\pi^{TM} \circ X_i^{\mathrm{C}} &= \left(\frac{\partial}{\partial x^i} + (-K_{ij}^a v^j - \Upsilon_{ib}^a w^b)\frac{\partial}{\partial w^a}\right) \circ \pi^{TM},\\
T\pi^{TM} \circ X_i^{\mathrm{V}} &= \frac{\partial}{\partial v^j} \circ \pi^{TM},\\
T\pi^{TM} \circ \hat{E}_a^{\mathrm{C}} &= (\Upsilon_{ia}^b v^i + C_{ac}^b w^c)\frac{\partial}{\partial w^b} \circ \pi^{TM},\\
T\pi^{TM} \circ \hat{E}_a^{\mathrm{V}} &= \frac{\partial}{\partial w^a} \circ \pi^{TM}.
\end{aligned}
$$

The Lagrangian field in case of symmetry

• Since Γ is a sode, it is of the form

$$\Gamma = w^a \hat{E}_a^{\rm C} + v^i X_i^{\rm C} + \Gamma^a \hat{E}_a^{\rm V} + \Gamma^i X_i^{\rm V},$$

It satisfies:

$$\left\{ \begin{array}{l} \Gamma(X_i^{\rm V}(L)) - X_i^{\rm C}(L) = 0, \\ \Gamma(\hat{E}_b^{\rm V}(L)) - \hat{E}_b^{\rm C}(L) = 0. \end{array} \right.$$

Reduction? L reduces to a function $\bar{L}$ on $TM/G \rightsquigarrow L = \bar{L} \circ \pi^{TM}$;

Γ reduces to a VF $\bar{\Gamma}$ on $TM/G \rightsquigarrow T\pi^{TM} \circ \Gamma = \bar{\Gamma} \circ \pi^{TM}$.

• Since all L, Γ, etc. are invariant, the defining relation for the reduced vector field is :

$$\left\{ \begin{array}{l} \bar{\Gamma}(\bar{X}_i^{\rm V}(\bar{L})) - \bar{X}_i^{\rm C}(\bar{L}) = 0, \\ \bar{\Gamma}(\bar{E}_b^{\rm V}(\bar{L})) - \bar{E}_b^{\rm C}(\bar{L}) = 0. \end{array} \right.$$

The defining relation for the reduced vector field is :

$$\begin{cases} \bar{\Gamma}(\bar{X}_i^{\mathrm{V}}(\bar{L})) - \bar{X}_i^{\mathrm{C}}(\bar{L}) = 0, \\ \bar{\Gamma}(\bar{E}_b^{\mathrm{V}}(\bar{L})) - \bar{E}_b^{\mathrm{C}}(\bar{L}) = 0. \end{cases}$$

with

$$\begin{aligned} \bar{E}_a^{\mathrm{C}} &= \left(\Upsilon_{ia}^b v^i + C_{ac}^b w^c\right) \frac{\partial}{\partial w^b}, \qquad \bar{E}_a^{\mathrm{V}} = \frac{\partial}{\partial w^a}, \\ \bar{X}_i^{\mathrm{C}} &= \frac{\partial}{\partial x^i} - \left(K_{ij}^a v^j + \Upsilon_{ib}^a w^b\right) \frac{\partial}{\partial w^b}, \qquad \bar{X}_i^{\mathrm{V}} = \frac{\partial}{\partial v^i}. \end{aligned}$$

- We have $\Gamma = w^a \hat{E}_a^{\mathrm{C}} + v^i X_i^{\mathrm{C}} + \Gamma^a \hat{E}_a^{\mathrm{V}} + \Gamma^i X_i^{\mathrm{V}}$.

Each term is invariant, so Γ^a and Γ^i define functions on TM/G. We have

$$\begin{aligned}\bar{\Gamma} &= w^a(\Upsilon^b_{ia} v^i + C^b_{ac} w^c)\frac{\partial}{\partial w^b} + v^i \frac{\partial}{\partial x^i} \\ &\quad - v^i \left(K^a_{ij} v^j + \Upsilon^a_{ib} w^b \right) \frac{\partial}{\partial w^b} + \Gamma^a \frac{\partial}{\partial w^a} + \Gamma^i \frac{\partial}{\partial v^i} \\ &= v^i \frac{\partial}{\partial x^i} + \Gamma^i \frac{\partial}{\partial v^i} + \Gamma^a \frac{\partial}{\partial w^a}.\end{aligned}$$

The reduced equations become

$$\bar{\Gamma}\left(\frac{\partial \bar{L}}{\partial v^i}\right) - \frac{\partial \bar{L}}{\partial x^i} = (K^a_{ik} v^k + \Upsilon^a_{ib} w^b)\frac{\partial \bar{L}}{\partial w^a}$$

$$\bar{\Gamma}\left(\frac{\partial \bar{L}}{\partial w^a}\right) = (\Upsilon^b_{ia} v^i + C^b_{ac} w^c)\frac{\partial \bar{L}}{\partial w^b}.$$

These are the **Lagrange-Poincaré equations**. They are completely determined by $\bar{L}$.

Wong's equations

Let g be a Riemannian metric on M

(1) on which a group G acts freely and properly to the left as isometries, $\mathcal{L}_{\tilde{\xi}} g = 0$,

(2) whose vertical part (that is, its restriction to the fibres of $\pi^M : M \to M/G$) comes from a bi-invariant metric on G.

We will use the mechanical connection: $g(\hat{E}_a, X_i) = 0$.

From the symmetry (1): the components

$$g(\hat{E}_a, \hat{E}_b) = h_{ab}, \quad g(X_i, X_j) = g_{ij}.$$

are invariant functions and pass to the quotient.

$\rightsquigarrow$ g_{ij} form a metric on M/G.

Assumption (2) means that

(a) $\mathcal{L}_{\hat{E}_c} g(\hat{E}_a, \hat{E}_b) = 0$ (as well as $\mathcal{L}_{\tilde{E}_c} g(\hat{E}_a, \hat{E}_b) = 0$)

(b) the h_{ab} must be independent of the coordinates x^i on M/G, which is to say that they must be constants.

From (a): h_{ab} must satisfy

$$h_{ad} C^d_{bc} + h_{bd} C^d_{ac} = 0.$$

If we set

$$X_i = \frac{\partial}{\partial x^i} - \gamma^a_i \hat{E}_a$$

for some G-invariant coefficients γ^a_i, we get $\Upsilon^b_{ia} = \gamma^c_i C^b_{ac}$, and therefore

$$h_{ac} \Upsilon^c_{ib} + h_{bc} \Upsilon^c_{ib} = 0.$$

Geodesic equations

The geodesic equations may be derived from the Lagrangian

$$L = \tfrac{1}{2} g_{\alpha\beta} u^\alpha u^\beta = \tfrac{1}{2} g_{ij} v^i v^j + \tfrac{1}{2} h_{ab} w^a w^b.$$

It is of course G-invariant.

We may therefore apply Lagrange-Poincaré reduction, which gives the reduced equations

$$\begin{aligned} \frac{d}{dt}(g_{ij} v^j) - \tfrac{1}{2} \frac{\partial g_{jk}}{\partial x^i} v^j v^k &= -(K_{ij}^a v^j + \Upsilon_{ib}^a w^b) h_{ac} w^c \\ \frac{d}{dt}(h_{ab} w^b) &= (\Upsilon_{ia}^b v^i + C_{ac}^b w^c) h_{bd} w^d. \end{aligned}$$

Since $\Upsilon_{ib}^a h_{ac}$ is skew in b and c, and $C_{ac}^b h_{bd}$ is skew in c and d, the final terms in each equation are zero.

Let Γ^i_{jk} be the connection coefficients of g_{ij}. Then,

$$\begin{aligned} g_{ij}\left(\ddot{x}^j + \Gamma^j_{kl}\dot{x}^k\dot{x}^l\right) &= -h_{bc}K^c_{ij}\dot{x}^j w^b \\ h_{ab}\left(\dot{w}^b + \Upsilon^b_{ic}\dot{x}^i w^c\right) &= 0, \end{aligned}$$

using the skew-symmetry of $\Upsilon^c_{ib}h_{ac}$ again in the second equation.

Given that K^c_{ij} is skew in i and j, we get

$$\begin{aligned} \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k &= g^{ik}h_{bc}K^c_{jk}\dot{x}^j w^b \\ \dot{w}^a + \Upsilon^a_{jb}\dot{x}^j w^b &= 0. \end{aligned}$$

These are **Wong's equations**.

A charged particle in a magnetic field

Consider $M = \mathrm{E}^3 \times \mathrm{S}$, with coordinates (x^i, θ).

Let A_i be the components of a covector field on E^3. The *Kaluza-Klein metric* g on M is

$$g = \delta_{ij} dx^i \odot dx^j + (A_i dx^i + d\theta)^2.$$

It admits the Killing field $E = \partial/\partial\theta$.

The vector fields $X_i = \partial/\partial x^i - A_i \partial/\partial\theta$ are orthogonal to E and invariant; moreover $g_{ij} = g(X_i, X_j) = \delta_{ij}$, while $g(E, E) = 1$.

Finally

$$[X_i, X_j] = \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right) \frac{\partial}{\partial\theta}.$$

Putting these values into the reduced equations above we obtain

$$\ddot{x}^i = w\dot{x}^j \left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i} \right), \qquad \dot{w} = 0.$$

These are the equations of motion of a particle of unit mass and charge w in a magnetic field whose vector potential is $A_i dx^i$.

A worked-out example

Consider G whose elements are affine maps

$$\mathbf{R} \to \mathbf{R} : t \mapsto \exp(\theta)t + \phi$$

or, represented as a matrix,

$$\begin{pmatrix} \exp\theta & \phi \\ 0 & 1 \end{pmatrix}.$$

The identity element is just $t \mapsto t$ (the identity matrix) and multiplication is the composition of the two affine maps, i.e.

$$(\theta_1, \phi_1) * (\theta_2, \phi_2) = (\theta_1 + \theta_2, \exp(\theta_1)\phi_2 + \phi_1).$$

Its Lie algebra is given by the set of matrices of the form

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

Consider on $M = G \times \mathbf{R}$ the action given by left translation on G. We use $x^0 = x$ for the coordinate on $\mathbf{R}$.

A basis of fundamental vector fields is

$$\tilde{E}_1 = \frac{\partial}{\partial\theta} + \phi\frac{\partial}{\partial\phi}, \qquad \tilde{E}_2 = \frac{\partial}{\partial\phi}.$$

We have $[\tilde{E}_1, \tilde{E}_2] = -\tilde{E}_2$. The complete and vertical lifts are

$$\tilde{E}_1^{\mathrm{C}} = \frac{\partial}{\partial\theta}+\phi\frac{\partial}{\partial\phi}+\dot{\phi}\frac{\partial}{\partial\dot{\phi}}, \quad \tilde{E}_2^{\mathrm{C}} = \frac{\partial}{\partial\phi}, \quad \tilde{E}_1^{\mathrm{V}} = \frac{\partial}{\partial\dot{\theta}}+\phi\frac{\partial}{\partial\dot{\phi}}, \quad \tilde{E}_2^{\mathrm{V}} = \frac{\partial}{\partial\dot{\phi}}.$$

We use the trivial principal connection with $\gamma_a^0 = 0$. Horizontal vector fields lie in the direction of $X = \partial/\partial x$.

Its quasi-velocities (v^i, v^a) are $v^0 = \dot{x}$ and $v^1 = \dot{\theta}$, $v^2 = \dot{\phi} - \phi\dot{\theta}$.

An invariant basis of vector fields is given by $\{\hat{E}_1, \hat{E}_2, X\}$, where

$$\hat{E}_1 = \frac{\partial}{\partial\theta}, \qquad \hat{E}_2 = \exp(\theta)\frac{\partial}{\partial\phi}.$$

Its quasi-velocities are $v^0 = \dot{x}$, $w_1 = \dot{\theta}$ and $w_2 = \exp(-\theta)\dot{\phi}$.

The lifts are

$$X^{\mathrm{C}} = \frac{\partial}{\partial x}, \quad \hat{E}_1^{\mathrm{C}} = \frac{\partial}{\partial\theta}, \quad \hat{E}_2^{\mathrm{C}} = \exp(\theta)\Big(\frac{\partial}{\partial\phi} + \dot{\theta}\frac{\partial}{\partial\dot{\phi}}\Big),$$
$$X^{\mathrm{V}} = \frac{\partial}{\partial\dot{x}}, \quad \hat{E}_1^{\mathrm{V}} = \frac{\partial}{\partial\dot{\theta}}, \quad \hat{E}_2^{\mathrm{V}} = \exp(\theta)\frac{\partial}{\partial\dot{\phi}}.$$

The matrix $\mathcal{A}$, defined by $\hat{E}_a(x,g) = A_a^b(g)\tilde{E}_b(x,g)$, is

$$A(g) = \left(\begin{array}{cc} 1 & 0 \\ -\phi & \exp(\theta) \end{array}\right).$$

At the identity of G, A is the identity matrix.

If we use the invariant coordinates (v^0, w^a), the action on TM is

$$\psi^{TM}_{(\phi_1,\theta_1)}(x,(\phi,\theta),\dot{x},w_1,w_2) = (x,(\phi_1,\theta_1)*(\phi,\theta),\dot{x},w_1,w_2).$$

The coordinates $(x,\dot{x},w_1,w_2)$ can be interpreted as coordinates on $TM/G = T\mathbf{R}\times TG/G = T\mathbf{R}\times\mathfrak{g}$.

We will work with the Lagrangian

$$L = \tfrac{1}{2}\dot{\theta}^2 + q\dot{x}\dot{\theta} + \tfrac{1}{2}\dot{x}^2 + \ln(\exp(-\theta)\dot{\phi}), \qquad q \text{ constant.}$$

This Lagrangian is invariant: $\tilde{E}_1^{\mathrm{C}}(L) = 0 = \tilde{E}_2^{\mathrm{C}}(L)$.

In the coordinates $(w_i = (w_1, w_2), \dot{x})$, the Lagrangian is

$$L = \tfrac{1}{2}w_1^2 + q\dot{x}w_1 + \tfrac{1}{2}\dot{x}^2 + \ln(w_2),$$

so, indeed, (ϕ,θ) do not appear explicitly.

The Hessian matrix in the basis $\{\tilde{E}_a, X\}$ is here

$$g = \begin{pmatrix} 1 - \frac{\phi^2}{\dot\phi^2} & -\frac{\phi}{\dot\phi^2} & q \\ -\frac{\phi}{\dot\phi^2} & -\frac{1}{\dot\phi^2} & 0 \\ q & 0 & 1 \end{pmatrix}.$$

$\rightsquigarrow$ Since $\det g = (q^2 - 1)/\dot\phi^2$, L is regular when $q^2 \neq 1$.

The upper left (2,2) matrix represents (g_{ab}).

$\rightsquigarrow$ It is non-singular since its determinant is $-1/\dot\phi^2$.

$\rightsquigarrow$ Its inverse is $(g^{ab}) = \begin{pmatrix} 1 & -\phi \\ -\phi & \phi^2 - \dot\phi^2 \end{pmatrix}$.

The vector field that is horizontal with respect to Ω^{m} is

$$X^{\mathrm{cm}} = \frac{\partial}{\partial x} - q\frac{\partial}{\partial \theta} - q\dot\phi\frac{\partial}{\partial \dot\phi}.$$

The Euler-Lagrange equations are:

$$q\ddot{\theta} + \ddot{x} = 0, \qquad \ddot{\theta} + q\ddot{x} + 1 = 0, \qquad -\frac{\ddot{\phi}}{\dot{\phi}^2} = 0.$$

We can solve them directly and obtain

$$x(t) = -\tfrac{1}{2}\frac{qt^2}{q^2-1} + \dot{x}_0 t + x_0,$$

$$\theta(t) = \tfrac{1}{2}\frac{t^2}{q^2-1} + \dot{\theta}_0 t + \theta_0, \qquad \phi(t) = \dot{\phi}_0 t + \phi_0.$$

We will assume that $\dot{\phi}_0 > 0$.

Solutions of the Lagrange-Poincaré equations

The reduced Lagrangian on TM/G is

$$\bar{L}(x, \dot{x}, w_1, w_2) = \tfrac{1}{2} w_1^2 + q\dot{x} w_1 + \tfrac{1}{2}\dot{x}^2 + \ln(w_2),$$

and the reduced equations are

$$\dot{w}_1 + q\ddot{x} = -1, \qquad \dot{w}_2 = -w_1 w_2, \qquad q\dot{w}_1 + \ddot{x} = 0.$$

If we set $w_1(0) = \dot{\theta}_0$ and $w_2(0) = \exp(-\theta_0)\dot{\phi}_0$, the solution of the above equations is

$$x(t) = -\tfrac{1}{2}\frac{qt^2}{q^2 - 1} + \dot{x}_0 t + x_0, \qquad w_1(t) = \frac{t}{q^2 - 1} + \dot{\theta}_0,$$

$$w_2(t) = \dot{\phi}_0 \exp\left(-\theta_0 - \frac{1}{q^2 - 1}(\tfrac{1}{2}t^2 - \dot{\theta}_0 t + \dot{\theta}_0 q^2 t)\right).$$

Horizontal lift of a curve

The coordinates $\bar{v}^{\mathrm{H}}(t) = (x(t), \phi^{\mathrm{H}}(t), \theta^{\mathrm{H}}(t), \dot{x}(t), w_1(t), w_2(t))$ of the horizontal lift in the invariant basis can be determined from

$$0 = \Omega^{\mathrm{m}}(\dot{\bar{v}}^{\mathrm{H}}(t)) = \omega^{\mathrm{m}}(\bar{v}^{\mathrm{H}}, T\tau \circ \dot{\bar{v}}^{\mathrm{H}}) = \omega^{\mathrm{m}}\left(\bar{v}^{\mathrm{H}}, \frac{d}{dt}(\tau \circ \bar{v}^{\mathrm{H}})\right).$$

This equation for $\tau \circ \bar{v}^{\mathrm{H}}(t) = (x(t), \phi^{\mathrm{H}}(t), \theta^{\mathrm{H}}(t))$ is

$$\dot{\theta}^{\mathrm{H}} = -q\dot{x}, \qquad \dot{\phi}^{\mathrm{H}} = -q\dot{x}\phi^{\mathrm{H}} + q\dot{x}\phi^{\mathrm{H}} = 0.$$

Therefore $\theta^{\mathrm{H}}(t) = -qx(t) + qx_0 + \theta_0$ and $\phi^{\mathrm{H}}(t) = \phi_0$.

Reconstruction equation

The curve $g(t) = (\theta_1(t), \phi_1(t))$ in G is such that $v = g\bar{v}^{\mathrm{H}}$ solves the Euler-Lagrange equations with the given initial values.

The reconstruction equation is $g^{-1}\dot{g} = \Omega^{\mathrm{m}}(\Gamma \circ \bar{v}^{\mathrm{H}})$.

The left hand side $\dot{\theta}_1 E_1 + \exp(-\theta_1)\dot{\phi}_1 E_2$.

The right hand side is $(w_1 + q\dot{x})\tilde{E}_1^{\mathrm{C}} \circ \bar{v}^{\mathrm{H}} + (w_2 + q\phi_m\dot{x})\tilde{E}_1^{\mathrm{C}} \circ \bar{v}^{\mathrm{H}}$.

The reconstruction equations are therefore

$$\dot{\theta}_1 = w_1 + q\dot{x}, \qquad \exp(-\theta_1)\dot{\phi}_1 = -\phi^{\mathrm{H}} w_1 + \exp(\theta_m) w_2 - q\phi^{\mathrm{H}}\dot{x}.$$

Solving the above equations for (θ_1, ϕ_1) gives

$$\begin{aligned}\theta_1(t) &= -\tfrac{1}{2}t^2 + (q\dot{x}_0 + \dot{\theta}_0)t, \\ \phi_1(t) &= \dot{\phi}_0 t + \phi_0\Big(1 - \exp(\tfrac{1}{2}(2q\dot{x}_0 - t + 2\dot{\theta}_0)t)\Big).\end{aligned}$$

The final solution is therefore (indeed)

$$\begin{aligned}
\theta(t) &= \theta_1(t) + \theta^{\mathrm{H}}(t) = \tfrac{1}{2}\frac{t^2}{q^2-1} + \dot{\theta}_0 t + \theta_0, \\
\phi(t) &= \exp(\theta_1(t))\phi^{\mathrm{H}}(t) + \phi_1(t) = \dot{\phi}_0 t + \phi_0.
\end{aligned}$$

3. Routh reduction

We use the frame $\{X_i, \tilde{E}_a\}$ on M, with quasi-velocities

$$v_m = v^i X_i(m) + v^a \tilde{E}_a(m).$$

We may determine Γ from

$$\begin{array}{rcl} \Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) & = & 0 \\ \Gamma(\tilde{E}_a^{\mathrm{V}}(L)) - \tilde{E}_a^{\mathrm{C}}(L) & = & 0. \end{array}$$

From $\tilde{E}_a^{\mathrm{C}}(L) = 0$ we get that $p_a = \tilde{E}_a^{\mathrm{V}}(L)$ are first integrals of Γ.

For $\mu \in \mathfrak{g}^*$, the subset

$$p_a = \mu_a \qquad \Leftrightarrow \qquad v^a = \iota^a(x^\alpha, v^i)$$

defines (under the assumed G-regularity) a submanifold $N_\mu \subset TM$.

Γ is tangent to each N_μ. Thus, $\Gamma_\mu = \Gamma\,|_{N_\mu}$ is a vector field on N_μ.

Symmetry

Let G_μ be the isotropy group of $\mu \in \mathfrak{g}^*$, $G_\mu = \{g \in G \mid \mathrm{ad}^*_g \mu = \mu\}$.

Lemma. The lifted action $G \times TM \to TM$ restricts to an action $G_\mu \times N_\mu \to N_\mu$.

Proof. (Infinitesimal version) For $\xi^{\mathrm{C}}_M = \xi^b \tilde{E}^{\mathrm{C}}_b$, we have

$$\begin{aligned} \xi^{\mathrm{C}}_M(p_a) &= \xi^b \tilde{E}^{\mathrm{C}}_b \left(\tilde{E}^{\mathrm{V}}_a L \right) = \xi^b [\tilde{E}_b, \tilde{E}_a]^{\mathrm{V}}(L) = -\xi^b C^d_{ba} \tilde{E}^{\mathrm{V}}_d(L) \\ &= -\xi^b C^d_{ba} p_d \end{aligned}$$

This will only vanish on N_μ when $\xi^b C^d_{ba} \mu_d = 0$ or $\xi \in \mathfrak{g}_\mu$. □

Proposition. (1) The vector field $\Gamma_\mu = \Gamma\,|_{N_\mu}$ on N_μ is G_μ-invariant. It reduces to a vector field on N_μ / G_μ.

(2) There is a version of the mechanical connection available for reconstruction.

A frame for vector fields on N_μ

We define vector fields which are tangent to N_μ.
Since (g_{ab}) is non-singular, there are uniquely defined coefficients A_i^b, B_i^b and C_a^b such that

$$
\begin{aligned}
(X_i^{\mathrm{C}} + A_i^b \tilde{E}_b^{\mathrm{V}})(p_a) &= X_i^{\mathrm{C}}(p_a) + A_i^b g_{ab} = 0 \\
(X_i^{\mathrm{V}} + B_i^b \tilde{E}_b^{\mathrm{V}})(p_a) &= X_i^{\mathrm{V}}(p_a) + B_i^b g_{ab} = 0 \\
(\tilde{E}_a^{\mathrm{C}} + C_a^b \tilde{E}_b^{\mathrm{V}})(p_c) &= \tilde{E}_a^{\mathrm{C}}(p_c) + C_a^b g_{bc} = 0.
\end{aligned}
$$

Define vector fields $\bar{X}_i^{\mathrm{C}}$, $\bar{X}_i^{\mathrm{V}}$ and $\bar{E}_a^{\mathrm{C}}$ by

$$
\begin{aligned}
\check{X}_i^{\mathrm{C}} &= X_i^{\mathrm{C}} + A_i^a \tilde{E}_a^{\mathrm{V}} \\
\check{X}_i^{\mathrm{V}} &= X_i^{\mathrm{V}} + B_i^a \tilde{E}_a^{\mathrm{V}} \\
\check{E}_a^{\mathrm{C}} &= \tilde{E}_a^{\mathrm{C}} + C_a^b \tilde{E}_b^{\mathrm{V}};
\end{aligned}
$$

they are tangent to each level set N_μ. Their restriction to N_μ gives a frame of vector fields on N_μ.

We have

$$\check{X}_i^{\mathrm{V}}(v^j) = \delta_i^j, \quad \check{X}_i^{\mathrm{C}}(v^j) = 0, \quad \check{E}_a^{\mathrm{C}}(v^i) = 0,$$
$$\check{X}_i^{\mathrm{V}}(v^a) = B_i^a, \quad \check{X}_i^{\mathrm{C}}(v^a) = -R_{ij}^a v^j + A_i^a, \quad \check{E}_a^{\mathrm{C}}(v^b) = C_{ac}^b v^c + C_a^b.$$

Lie brackets: $[\check{E}_a^{\mathrm{C}}, \check{X}_i^{\mathrm{V}}] = 0$, $[\check{X}_i^{\mathrm{C}}, \check{X}_j^{\mathrm{V}}] = 0$, etc.

Given that $T\tau \circ \Gamma = \mathrm{id}$, we see that

$$\Gamma_\mu = v^a \check{E}_a^{\mathrm{C}} + v^i \check{X}_i^{\mathrm{C}} + \Gamma^i \check{X}_i^{\mathrm{V}} = \iota^a \check{E}_a^{\mathrm{C}} + v^i \check{X}_i^{\mathrm{C}} + \Gamma^i \check{X}_i^{\mathrm{V}}.$$

What is Γ^i? It should be determined from

$$\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0.$$

Routh equations

We will rewrite $\Gamma(X_i^{\mathrm{V}}(L)) - X_i^{\mathrm{C}}(L) = 0$ in terms of the Routhian

$$\mathcal{R} = L - v^a p_a.$$

We have

$$\begin{aligned} X_i^{\mathrm{C}}(L) &= \check{X}_i^{\mathrm{C}}(L) - A_i^a \tilde{E}_a^{\mathrm{V}}(L) \\ &= \check{X}_i^{\mathrm{C}}(L - v^a p_a) + (-R_{ij}^a v^j + A_i^a) p_a + v^a \check{X}_i^{\mathrm{C}}(p_a) - A_i^a p_a \\ &= \check{X}_i^{\mathrm{C}}(\mathcal{R}) - p_a R_{ij}^a v^j; \\ X_i^{\mathrm{V}}(L) &= \check{X}_i^{\mathrm{V}}(L) - B_i^a \tilde{E}_a^{\mathrm{V}}(L) \\ &= \check{X}_i^{\mathrm{V}}(L - v^a p_a) + B_i^a p_a + v^a \check{X}_i^{\mathrm{V}}(p_a) - B_i^a p_a \\ &= \check{X}_i^{\mathrm{V}}(\mathcal{R}). \end{aligned}$$

Thus, if $\mathcal{R}^\mu$ is the restriction to N_μ, we get

$$\Gamma(\check{X}_i^{\mathrm{V}}(\mathcal{R}^\mu)) - \check{X}_i^{\mathrm{C}}(\mathcal{R}^\mu) = -\mu_a R_{ij}^a v^j.$$

These are the Routh equations.

$$\Gamma(\check{X}_i^{\mathrm{V}}(\mathcal{R}^\mu)) - \check{X}_i^{\mathrm{C}}(\mathcal{R}^\mu) = -\mu_a R^a_{ij} v^j .$$

Now all the involved vector fields are tangent to N_μ and can be dropped to N_μ / G_μ...

We know

$$\Gamma = v^a \check{E}_a^{\mathrm{C}} + v^i \check{X}_i^{\mathrm{C}} + \Gamma^i \check{X}_i^{\mathrm{V}} .$$

The matrix

$$\check{X}_i^{\mathrm{V}}(\check{X}_j^{\mathrm{V}}(\mathcal{R})) = (X_i^{\mathrm{V}} - g^{ab} g_{ib} \tilde{E}_a^{\mathrm{V}})(X_j^{\mathrm{V}}(L)) = g_{ij} - g^{ab} g_{ia} g_{jb} .$$

is under the stated conditions always non-singular.

The components Γ^i can therefore be obtained from the Routh equations.

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