UNIVERSIDAD DE LA LAGUNA

Departamento de Matemática Fundamental

Sección de Geometría y Topología

GRUPOS Y GRUPOIDES DE LIE
Y
ESTRUCTURAS DE JACOBI

David Iglesias Ponte

Memoria realizada bajo la dirección del Profesor Dr. D. Juan Carlos Marrero González para optar al grado de doctor por la Universidad de La Laguna

La Laguna, Abril 2003

Agradecimientos

En primer lugar, me gustaría dar las gracias al Prof. Juan Carlos Marrero González, director de esta Tesis, por toda la energía y tiempo dedicados a que ésta haya llegado a buen puerto.

Por otra parte, agradezco a la Prof. Edith Padrón Fernández por toda la ayuda que me ha brindado en el tiempo que ha durado la realización de esta Tesis. Los Prof. Ángel Montesdeoca Delgado y Josué Remedios Gómez han tenido mucha paciencia conmigo a la hora de concederme su tiempo y conocimientos informáticos en la elaboración de este texto.

A los miembros del Workshop Differential Geometric Methods in Theoretical Mechanics, por el buen ambiente que he disfrutado en todos los encuentros a los que he asistido.

Quiero agradecer a toda mi familia, particularmente a mis padres y mi hermana, por todo su apoyo y comprensión en este largo camino. Gracias a ellos, estoy hoy aquí.

Finalmente, dedico con todo mi amor esta Tesis a María, por su presencia cálida y constante en los momentos buenos y los no tan buenos por los que he pasado.

Indice

Resumen

Antecedentes y estado actual del tema

La noción de variedad de Poisson fue introducida por Lichnerowicz en [73] (ver también [115]). Una estructura de Poisson en una variedad M es un corchete de álgebra de Lie en el espacio de las funciones reales C^∞-diferenciables en M, $C^\infty(M,\mathbb{R})$, tal que es una derivación en cada uno de los argumentos con respecto al producto usual de funciones. Una de las principales motivaciones para la introducción de esta noción es que las variedades de Poisson juegan un papel importante en la Mecánica Clásica. De hecho, los corchetes de Poisson aparecen de manera natural en el estudio de algunos sistemas mecánicos, particularmente sistemas con ligaduras o en la reducción de sistemas con grupos de simetría. Pero la geometría de Poisson es también relevante para las álgebras de observables en la Mecánica Cuántica. De hecho, Kontsevich [59] ha mostrado que la clasificación de deformaciones formales del álgebra $C^\infty(M,\mathbb{R})$ para cualquier variedad M es equivalente a la clasificación de familias formales de estructuras de Poisson en M.

Geométricamente, el corchete de Poisson induce un 2-vector Π en M, caracterizado por la relación $\{f,g\} = \Pi(d_0 f, d_0 g)$, para $f,g \in C^\infty(M,\mathbb{R})$, donde

d_0 es la diferencial exterior sobre M. Así, la identidad de Jacobi para $\{\, ,\, \}$ puede ser reinterpretada como la condición

$$[\Pi, \Pi] = 0,$$

donde $[\, ,\,]$ es el corchete de Schouten-Nijenhuis (ver [3, 73, 110]).

Dos ejemplos interesantes de variedades de Poisson son las variedades simplécticas y las estructuras de Lie-Poisson en el dual de un álgebra de Lie. De hecho, una variedad de Poisson está hecha de piezas simplécticas en el sentido de que admite una foliación generalizada, la foliación simpléctica, cuyas hojas son variedades simplécticas.

Otra categoría con una relación cercana a la geometría Poisson es la de los algebroides de Lie. Un algebroide de Lie sobre una variedad M es un fibrado vectorial A sobre M tal que su espacio de secciones $\Gamma(A)$ admite un corchete de álgebra de Lie $[\![\, ,\,]\!]$ y, además, existe una aplicación fibrada ρ de A en TM, la aplicación ancla, tal que el correspondiente homomorfismo de $C^\infty(M, \mathbb{R})$-módulos, también denotado por $\rho : \Gamma(A) \to \mathfrak{X}(M)$, satisface una relación de tipo Leibniz, esto es,

$$[\![X, fY]\!] = \rho(X)(f)Y + f[\![X, Y]\!],$$

para $X, Y \in \Gamma(A)$ y $f \in C^\infty(M, \mathbb{R})$ (ver [82, 99]). Los algebroides de Lie son una generalización natural de los fibrados tangentes y de las álgebras de Lie reales de dimensión finita. Pero existen muchos otros ejemplos interesantes. Así, la estructura de álgebra de Lie en el espacio de las funciones reales C^∞-diferenciables en una variedad de Poisson M nos permite definir un corchete de Lie en el espacio de las 1-formas, el cual dota al fibrado cotangente T^*M con una estructura natural de algebroide de Lie ([3, 14, 30, 65, 110]). Existe también otra conexión entre las variedades de Poisson y los algebroides de Lie: Hay una correspondencia biyectiva entre estructuras de algebroide de Lie en un fibrado vectorial $\tau : A \to M$ y estructuras de Poisson homogéneas en el fibrado dual A^* (ver [14, 15]). En el caso particular en el que M es un punto, esto es, A es un álgebra de Lie real de dimensión finita, la correspondiente estructura de Poisson homogénea en A^* es justamente la estructura de Lie-Poisson usual.

Dos operadores importantes asociados con cualquier algebroide de Lie son el corchete de Schouten y la diferencial (ver [82]). Además, ciertas definiciones y construcciones relacionadas con álgebras de Lie graduadas, levantamientos de campos tensoriales sobre una variedad y estructuras de Poisson pueden ser generalizadas a algebroides de Lie arbitrarios (ver [36, 37]). En la misma dirección, en [38] los autores introdujeron la noción de algebroide como una extensión de la definición de algebroide de Lie y mostraron que muchos objetos del cálculo diferencial en una variedad (asociado con la estructura de algebroide de Lie canónica en TM) pueden ser obtenidos en el contexto de un algebroide general.

Desde el punto de vista de la Física, los algebroides de Lie pueden ser usados para dar descripciones geométricas de la Mecánica Lagrangiana y Hamiltoniana. Así, en [89], Martínez desarrolló una descripción geométrica de la Mecánica Lagrangiana independiente del tiempo sobre algebroides de Lie de una manera paralela al formalismo usual de Mecánica Lagrangiana en el fibrado tangente de una variedad. Otros artículos que estudian, en particular, diversos aspectos de sistemas Lagrangianos en algebroides de Lie son [8, 9, 13, 71, 119]. Más recientemente, otros autores (ver [31, 91, 103]) han comenzado una investigación sobre la posible generalización del concepto de algebroide de Lie a fibrados afines. La principal motivación fue crear un modelo geométrico que proporcione un entorno natural para una versión dependiente del tiempo de las ecuaciones de Lagrange en algebroides de Lie.

Por otra parte, los algebroides de Lie pueden ser considerados como los invariantes infinitesimales de los grupoides de Lie. Para ser precisos, recordamos primero que una categoría pequeña G sobre una base M es un conjunto G equipado con aplicaciones "source" y "target" de G en M, una sección unidad $\epsilon : M \to G$, y una multiplicación $(g,h) \mapsto gh$ definida en el conjunto $G^{(2)} = \{(g,h) \in G \times G \,/\, \alpha(g) = \beta(h)\}$ de pares componibles. Estas operaciones satisfacen las condiciones $\alpha(gh) = \alpha(h)$, $\beta(gh) = \beta(g)$, $(gh)k = g(hk)$ cuando alguno de los dos miembros está definido, y los elementos de $\epsilon(M)$ en G actúan como neutros para la multiplicación. Si todos los elementos de G poseen inverso con respecto a estas identidades, se dice que G es un grupoide. Si G y M son variedades, y las aplicaciones de estructura son diferenciables (se requiere que α y β sean sumersiones para asegurar que el dominio de la multiplicación es una variedad), entonces G se denomina un grupoide de Lie (ver, por ejemplo, [82]).

El álgebra de Lie de campos de vectores en un grupoide de Lie G contiene una subálgebra distinguida $\mathfrak{X}_L(G)$ de campos que son invariantes a izquierda en un cierto sentido; éstos son secciones de un fibrado vectorial AG el cual puede ser identificado con el fibrado normal a $\epsilon(M)$ en G y, entonces, con el núcleo de $T\beta$. $T\alpha$ es una aplicación ancla $AG \to TM$ para una estructura de algebroide de Lie en AG. A éste se le denomina el algebroide de Lie del grupoide de Lie G. Esta construcción generaliza la manera de obtener el álgebra de Lie asociada con cualquier grupo de Lie. Además, sabemos que cualquier álgebra de Lie puede ser integrada a un grupo de Lie conexo y simplemente conexo. Sin embargo, esto no es cierto para algebroides de Lie y grupoides de Lie: No todos los algebroides de Lie pueden ser integrados a grupoides de Lie (ver [82]). Recientemente, Crainic y Fernandes [18] han obtenido las obstrucciones precisas para integrar un algebroide de Lie arbitrario a un grupoide de Lie.

Hemos visto que el fibrado cotangente T^*M de una variedad de Poisson tiene una estructura de algebroide de Lie natural derivada del corchete de Poisson de funciones. Si existe un grupoide de Lie G cuyo algebroide de Lie es isomorfo a T^*M, decimos que M es una variedad de Poisson integrable. En este caso, existe una estructura simpléctica Ω en G para la cual el grafo de la multiplicación $\{(g,h,gh) \in G \times G \times G \,/\, \alpha(g) = \beta(h)\}$ es una subvariedad lagrangiana del producto simpléctico $(G\times G\times G, \Omega\oplus\Omega\oplus -\Omega)$. Un grupoide de Lie $G \rightrightarrows M$ dotado con una estructura simpléctica satisfaciendo esta propiedad se denomina un grupoide simpléctico (ver [14]). El espacio base de un grupoide simpléctico es siempre una variedad de Poisson. Un ejemplo canónico de grupoide simpléctico es el fibrado cotangente T^*G de un grupoide de Lie arbitrario $G \rightrightarrows M$, donde la estructura simpléctica es justamente la estructura simpléctica canónica Ω_{T^*G}. En este caso, el espacio base es A^*G y la estructura de Poisson en A^*G es justamente la estructura de Poisson lineal inducida por el algebroide de Lie AG. Una propiedad interesante es que la aplicación "source" de un grupoide simpléctico $\alpha : G \to M$ es un morfismo de Poisson. Realizaciones globales para variedades de Poisson arbitrarias fueron descubiertas por primera vez por Karasaev [55] y Weinstein [116] (resultados recientes pueden ser encontrados en [10, 19]).

Como acabamos de decir, los grupoides simplécticos aparecieron en los años 80 en los trabajos independientes de Karasaev [55] y Weinstein [116] (ver también los trabajos de Zakrzewski [122]), motivados por problemas de cuantificación. Mientras tanto, una teoría de grupos de Lie-Poisson había sido desarrollada a través de los trabajos de Drinfeld [27] y Semenov-Tian-Shansky [104, 105] sobre sistemas completamente integrables y grupos cuánticos (ver también [28, 63, 80, 81]). Nótese que un grupo de Lie-Poisson abeliano conexo y simplemente conexo es isomorfo al espacio dual de un álgebra de Lie real dotada con la estructura de Lie-Poisson. Por tanto, fue natural unificar la teoría de grupos de Lie-Poisson y la teoría de grupoides simplécticos. Para este propósito, Weinstein [117] introdujo la noción de grupoide de Poisson. Un grupoide de Poisson es un grupoide de Lie $G \rightrightarrows M$ con una estructura de Poisson Π para la cual el grafo de la multiplicación parcial es una subvariedad coisótropa en la variedad de Poisson $(G \times G \times G, \Pi \oplus \Pi \oplus -\Pi)$. Si $(G \rightrightarrows M, \Pi)$ es un grupoide de Poisson entonces existe una estructura de Poisson en M tal que la proyección "source" $\alpha : G \to M$ es un morfismo de Poisson. Además, si AG es el algebroide de Lie de G entonces el fibrado dual A^*G a AG también posee una estructura de algebroide de Lie. Por otra parte, los grupoides de Poisson tienen interesantes aplicaciones en la ecuación de Yang-Baxter dinámica clásica (ver, por ejemplo, [29, 68]).

En [83], Mackenzie y Xu probaron que un grupoide de Lie $G \rightrightarrows M$ dotado con una estructura de Poisson Π es un grupoide de Poisson si y sólo si la aplicación fibrada $\#_\Pi : T^*G \to TG$ inducida por Π es un morfismo entre el grupoide cotangente $T^*G \rightrightarrows A^*G$ y el grupoide tangente $TG \rightrightarrows TM$. Esta caracterización fue usada para probar que los bialgebroides de Lie son los invariantes infinitesimales de los grupoides de Poisson, esto es, si $(G \rightrightarrows M, \Pi)$ es un grupoide de Poisson entonces (AG, A^*G) es un bialgebroide de Lie y, recíprocamente, una estructura bialgebroide de Lie en el algebroide de Lie de un grupoide de Lie (α-simplemente conexo) puede ser integrada a una estructura de grupoide de Poisson [79, 83, 85] (estos resultados pueden ser aplicados para obtener una nueva demostración de un teorema de Karasaev [55] y Weinstein [116]). Un bialgebroide de Lie es un algebroide de Lie A tal que el fibrado vectorial dual A^* también posee una estructura de algebroide de Lie la cual es compatible con la de A en cierta manera (ver [83]). Los bialgebroides de Lie generalizan las biálgebras de Lie de Drinfeld [27]. Otro ejemplo importante de bialgebroide de Lie es el asociado con una estructura de Poisson. De forma más precisa, si M es una variedad de Poisson con 2-vector de Poisson Π y en TM (respectivamente, T^*M) consideramos la estructura de algebroide de Lie trivial (respectivamente, la estructura de algebroide de Lie cotangente asociada con Π) entonces el par (TM, T^*M) es un bialgebroide de Lie. La condición de compatibilidad de un bialgebroide de Lie ha sido expresada en el lenguaje de las álgebras de Gerstenhaber en [61].

Aunque las estructuras simpléctica y de Lie-Poisson son de Poisson, existen estructuras interesantes para la Mecánica Clásica, como las estructuras de contacto, que no lo son. Una generalización tanto de las variedades de Poisson como de las variedades de contacto, son las variedades de Jacobi. Una estructura de Jacobi en una variedad M es un 2-vector Λ y un campo de vectores E en M tal que $[\Lambda, \Lambda] = 2E \wedge \Lambda$ y $[E, \Lambda] = 0$ (ver [74]). Si (M, Λ, E) es una variedad de Jacobi uno puede definir un corchete de funciones, el corchete de Jacobi, de tal manera que el espacio $C^\infty(M, \mathbb{R})$ dotado con el corchete de Jacobi es un álgebra de Lie local en el sentido de Kirillov [57]. Recíprocamente, una estructura de álgebra de Lie local en $C^\infty(M, \mathbb{R})$ induce una estructura de Jacobi en M [39, 57]. Algunos ejemplos interesantes de variedades de

Jacobi, aparte de las variedades de Poisson y de contacto, son las variedades localmente conforme simplécticas (l.c.s.). De hecho, una variedad de Jacobi admite una foliación generalizada, llamada la foliación característica, cuyas hojas son variedades de contacto o l.c.s. (ver [24, 39, 57]).

Existe también una relación entre las variedades de Jacobi y los algebroides de Lie. De hecho, si M es una variedad arbitraria entonces el fibrado vectorial $TM \times \mathbb{R} \to M$ posee una estructura de algebroide de Lie natural. Además, si M es una variedad de Jacobi entonces el fibrado de 1-jets $T^*M \times \mathbb{R} \to M$ admite una estructura de algebroide de Lie [56] (para una variedad de Jacobi el fibrado vectorial T^*M no es, en general, un algebroide de Lie). Sin embargo, el pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ no es, en general, un bialgebroide de Lie (ver [111]).

La existencia de una estructura de algebroide de Lie asociada con cualquier estructura de Jacobi justifica la introducción de la noción de grupoide de contacto. Un grupoide de contacto $(G \rightrightarrows M, \eta, \sigma)$ es un grupoide de Lie $G \rightrightarrows M$ dotado con una 1-forma de contacto $\eta \in \Omega^1(G)$ y una función $\sigma \in C^\infty(G, \mathbb{R})$ tal que

$$\eta(gh)(X_g \oplus_{TG} Y_h) = \eta(g)(X_g) + e^{\sigma(g)}\eta(h)(Y_h),$$

para $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$, donde $\oplus_{TG}$ es la multiplicación parcial en el grupoide de Lie tangente $TG \rightrightarrows TM$ (ver [22, 23, 56, 69, 70]). Los grupoides de contacto pueden ser considerados como la versión en dimensión impar de los grupoides simplécticos y tienen aplicación en la precuantificación de variedades de Poisson y en la integración de estructuras de álgebras de Lie locales en fibrados vectoriales de rango 1 (ver [22, 23]). En este caso, la función σ es multiplicativa y el espacio base M posee una estructura de Jacobi de tal manera que el par (α, e^σ) es un morfismo conforme de Jacobi. Así, podemos considerar el algebroide de Lie $T^*M \times \mathbb{R} \to M$. De hecho, el algebroide de Lie AG de G es isomorfo a $T^*M \times \mathbb{R} \to M$.

Objetivos de la investigación y metodología

Como se indica en el título, el propósito general de esta Memoria es estudiar la relación que existe entre la teoría de grupoides de Lie (y algebroides de Lie) y las variedades de Jacobi.

Un esquema general de esta Memoria es el siguiente:

- Capítulo 1: Estructuras de Jacobi, algebroides y grupoides de Lie

  Este es un capítulo introductorio que contiene algunas generalidades sobre estructuras de Jacobi, algebroides de Lie y grupoides de Lie, tales como su definición, algunos ejemplos y propiedades que van a ser útiles a lo largo de la Memoria.

- Capítulo 2: Algebroides de Jacobi, estructuras de Jacobi homogéneas y su foliación característica

  Una estructura de algebroide de Jacobi en un fibrado vectorial es una estructura de algebroide Lie y un 1-cociclo en ella. En este capítulo, consideramos estructuras de Jacobi homogéneas en fibrados vectoriales. Obtenemos una caracterización de este tipo de estructuras y su relación con los algebroides de Jacobi. Finalmente, probamos que las hojas de la foliación característica de una estructura de Jacobi homogénea en un espacio vectorial son las órbitas de una acción de un grupo de Lie sobre el espacio vectorial y describimos dicha acción.

- Capítulo 3: Estructuras de Jacobi y bialgebroides de Jacobi

  Después de desarrollar un cálculo diferencial para los algebroides de Jacobi, introducimos la noción de bialgebroide de Jacobi (una generalización de la noción de bialgebroide de Lie) de tal manera que una variedad de Jacobi tiene asociado un bialgebroide de Jacobi canónico. De manera recíproca, probamos que se puede definir una estructura de Jacobi en el espacio base de un bialgebroide de Jacobi. También mostramos que es posible construir un bialgebroide de Lie desde un bialgebroide de Jacobi y, como consecuencia, deducimos un teorema de dualidad. La definición de un bialgebroide de Jacobi es ilustrada con varios ejemplos. En la última parte del capítulo, obtenemos una caracterización de los bialgebroides de Jacobi en términos de morfismos de algebroides de Jacobi.

- Capítulo 4: Biálgebras de Jacobi

  Estudiamos en este capítulo bialgebroides de Jacobi para los que el espacio base es un punto, esto es, biálgebras de Jacobi. Proponemos un método, que generaliza el método de la ecuación de Yang-Baxter, para obtener biálgebras de Jacobi y damos ejemplos de ellas. Finalmente, describimos las biálgebras de Jacobi compactas.

- Capítulo 5: Grupoides de Jacobi y bialgebroides de Jacobi

  Finalizamos la Memoria introduciendo los grupoides de Jacobi como una generalización de los grupoides de Poisson y de contacto. Se prueba que los bialgebroides de Jacobi son los invariantes infinitesimales de los grupoides de Jacobi.

Resultados obtenidos y conclusiones

Precisaremos ahora el contenido de cada uno de los capítulos incluidos en esta Memoria.

CAPITULO 1

Comenzamos esta memoria con el Capítulo 1, donde recordamos algunas definiciones y resultados sobre estructuras de Jacobi, algebroides y grupoides de Lie. En primer lugar, recordamos la definición de variedad de Jacobi y consideramos algunos ejemplos, incluyendo a las variedades de Poisson, así como otros ejemplos interesantes de variedades de Jacobi que no son Poisson, como las variedades de contacto o las localmente conforme simplécticas (ver Sección 1.1.2). Estas últimas estructuras son especialmente importantes ya que, de manera poco precisa, podemos decir que toda variedad de Jacobi está compuesta por piezas de contacto y localmente conforme simplécticas. De una forma más rigurosa, tenemos que las hojas de la foliación característica de una variedad de Jacobi son variedades de contacto o localmente conforme simplécticas (ver Sección 1.1.3).

Las estructuras de Poisson son ejemplos de estructuras de Jacobi, pero existe otra relación entre estructuras de Poisson y de Jacobi. De hecho, si M es una variedad de Jacobi entonces la variedad producto $M \times \mathbb{R}$ admite una estructura de Poisson exacta que se denomina la Poissonización de M. En la Sección 1.1.6 de este Capítulo, damos una descripción de esta estructura en $M \times \mathbb{R}$.

Es bien conocido que si M es una variedad de Jacobi entonces el fibrado vectorial $T^*M \times \mathbb{R} \to M$ admite una estructura de algebroide de Lie [56]. En la segunda parte de este Capítulo (Sección 1.2), recordamos la definición de estructura de algebroide de Lie en un fibrado vectorial A sobre una variedad M y la definición de dos operadores importantes asociados con cualquier algebroide de Lie: el corchete de Schouten de dos multi-secciones de A y la diferencial de una multi-sección del fibrado dual A^* de A. La diferencial es un operador de cohomología e induce el complejo de cohomología de algebroide de Lie con coeficientes triviales. Varios ejemplos de algebroides de Lie son considerados en la Sección 1.2.2, describiendo los elementos asociados con cada uno de ellos. En particular, presentamos la relación entre las estructuras de algebroide de Lie en un fibrado vectorial $\tau : A \to M$ y las estructuras de Poisson en el fibrado dual $\tau^* : A^* \to M$ que son homogéneas con respecto a Δ_{A^*}, el campo de Liouville de A^*.

Los objetos globales correspondientes a los algebroides de Lie son los grupoides de Lie. En la última Sección del Capítulo 1, recordamos la definición de un grupoide de Lie y de morfismo entre grupoides de Lie. Como en el caso de un grupo de Lie, uno puede considerar multi-vectores invariantes a izquierda en un grupoide de Lie. En particular, los campos de vectores invariantes a izquierda son cerrados con respecto al corchete de Schouten-Nijenhuis y pueden ser identificados con las secciones de un fibrado vectorial

$AG \to M$. Estos hechos permiten la construcción de una estructura de algebroide de Lie en AG. Finalmente, algunos ejemplos de grupoides de Lie son considerados en la Sección 1.3.2, describiendo el algebroide de Lie asociado en cada caso.

CAPITULO 2

En el Capítulo 1, hemos recordado la correspondencia biyectiva entre las estructuras de algebroide de Lie en un fibrado vectorial $\tau : A \to M$ y las estructuras de Poisson homogéneas en el fibrado dual A^*. Además, mostramos la relación entre la homogeneidad de una estructura de Poisson Π y el comportamiento del corchete de Poisson $\{\, , \,\}_\Pi$ con respecto a las funciones lineales. De hecho, una estructura de Poisson en un fibrado vectorial es homogénea si y sólo si la estructura de Poisson es lineal, es decir, las funciones lineales son cerradas con respecto al corchete de Poisson. En el Capítulo 2 de esta Memoria extendemos esta relación al contexto Jacobi. Más precisamente, en la Sección 2.1, describimos las estructuras de Jacobi homogéneas en un fibrado vectorial, esto es, estructuras de Jacobi (Λ, E) en un fibrado vectorial $A \to M$ tales que Λ y E son homogéneos con respecto al campo de Liouville de A. Además, como en el caso Poisson, deducimos la relación entre esta homogeneidad y el comportamiento del corchete de Jacobi $\{\, , \,\}_{(\Lambda,E)}$ entre funciones lineales y constantes. En particular, probamos que el campo E es el levantamiento vertical de cierta sección de $A \to M$ y que existe una estructura de Poisson homogénea Π_A tal que $\Lambda = \Pi_A + E \wedge \Delta_A$, siendo Δ_A el campo de Liouville en A (ver Teorema 2.3).

En la Sección 2.2, mostramos la relación entre las estructuras de Jacobi homogéneas en A^* y las estructuras de algebroide de Lie en A. De hecho, si (Λ, E) es una estructura de Jacobi homogénea en A^* entonces obtenemos que induce no sólo una estructura de algebroide de Lie en A, sino también un 1-cociclo $\phi_0 \in \Gamma(A^*)$ en el complejo de cohomología de A con coeficientes triviales. El campo E es, salvo el signo, el levantamiento vertical $\phi_0^{\mathbf{v}} \in \mathfrak{X}(A^*)$ de ϕ_0, esto es, $E = -\phi_0^{\mathbf{v}}$. Motivados por este resultado, introducimos la noción de algebroide de Jacobi como un par formado por una estructura de algebroide de Lie y un 1-cociclo en él. Después de mostrar un recíproco de este resultado, esto es, obtener una estructura de Jacobi homogénea en $A^* \to M$ desde una estructura de algebroide de Jacobi $((\llbracket\, , \rrbracket, \rho), \phi_0)$ en A, presentamos algunos ejemplos y aplicaciones en la Sección 2.3. Dos ejemplos interesantes son: i) para una variedad arbitraria M, el algebroide de Lie $A = TM \times \mathbb{R}$ y el 1-cociclo $\phi_0 = (0, -1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A^*)$, probamos que la estructura de Jacobi homogénea resultante en el fibrado vectorial $T^*M \times \mathbb{R} \to M$ es justamente la estructura de contacto canónica η_M; y ii) para una variedad de Jacobi (M, Λ, E), el algebroide de Lie $A^* = T^*M \times \mathbb{R}$ y el 1-cociclo $X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A)$, deducimos que la correspondiente estructura de Jacobi homogénea en el fibrado vectorial $TM \times \mathbb{R} \to M$ está dada por

$$\Lambda_{(TM\times\mathbb{R},X_0)} = \Lambda^c + \frac{\partial}{\partial t} \wedge E^c - t\Big(\Lambda^{\mathbf{v}} + \frac{\partial}{\partial t} \wedge E^{\mathbf{v}}\Big), \quad E_{(TM\times\mathbb{R},X_0)} = E^{\mathbf{v}},$$

donde Λ^c y E^c (resp., $\Lambda^{\mathbf{v}}$ y $E^{\mathbf{v}}$) es el levantamiento completo (resp. vertical) a TM de Λ y E, respectivamente. Esta estructura de Jacobi fue introducida por vez primera en [43] y es la versión Jacobi de la estructura de Poisson tangente usada por primera vez en [102] (ver también [15, 35]). Como aplicación de nuestra construcción, obtenemos una estructura de algebroide de Lie $(\llbracket\, , \rrbracket^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$ en el fibrado vectorial $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ desde una estructura de algebroide de Jacobi $((\llbracket\, , \rrbracket, \rho), \phi_0)$ en $A \to M$.

En la última Sección del Capítulo 2 (Sección 2.4), probamos que las hojas de la foliación característica de una estructura de Jacobi homogénea en el espacio dual $\mathfrak{g}^*$ de un espacio vectorial real $\mathfrak{g}$ son las órbitas de cierta acción de un grupo de Lie conexo y simplemente conexo $\tilde{G}$ sobre $\mathfrak{g}^*$ y describimos la estructura de Jacobi inducida en cada hoja. Como consecuencia, deducimos un resultado bien conocido: si $\Pi_{\mathfrak{g}^*}$ es una estructura de Poisson lineal en $\mathfrak{g}^*$ entonces $\mathfrak{g}$ es un álgebra de Lie y las hojas de la foliación simpléctica de $\Pi_{\mathfrak{g}^*}$ son las órbitas de la representación coadjunta asociada con un grupo de Lie conexo y simplemente conexo G con álgebra de Lie $\mathfrak{g}$.

CAPITULO 3

Motivados por los resultados obtenidos en el Capítulo 2, introducimos, en el Capítulo 3, un cálculo diferencial asociado con cualquier algebroide de Jacobi. Si $(A, ([\![\, , \,]\!], \rho), \phi_0)$ es un algebroide de Jacobi sobre M entonces la representación usual del álgebra de Lie $\Gamma(A)$ sobre el espacio $C^\infty(M, \mathbb{R})$ dada por la aplicación ancla ρ puede ser modificada y se obtiene una nueva representación. El operador de cohomología resultante d^{ϕ_0} se denomina la ϕ_0-diferencial de A. La ϕ_0-diferencial de A nos permite definir, de manera natural, la ϕ_0-derivada de Lie por una sección $X \in \Gamma(A)$, $\mathcal{L}_X^{\phi_0} = d^{\phi_0} \circ i_X + i_X \circ d^{\phi_0}$. Por otra parte, imitando la definición de corchete de Schouten de dos operadores diferenciales multilineales de primer orden en el espacio de las funciones reales C^∞-diferenciables en una variedad N (see [3]), introducimos el ϕ_0-corchete de Schouten de una k-sección P y una k'-sección P' como la $(k + k' - 1)$-sección dada por

$$[\![P, P']\!]^{\phi_0} = [\![P, P']\!] + (-1)^{k+1}(k-1)P \wedge (i_{\phi_0} P') - (k'-1)(i_{\phi_0} P) \wedge P',$$

donde $[\![\, , \,]\!]$ es el corchete de Schouten usual en A. Para estos operadores, describimos algunas de sus propiedades.

Por otra parte, si M es una variedad de Jacobi entonces, como sabemos, el fibrado de 1-jets $T^*M \times \mathbb{R} \to M$ admite una estructura de algebroide de Lie [56]. Sin embargo, si en el fibrado vectorial $TM \times \mathbb{R} \to M$ consideramos la estructura de algebroide de Lie natural entonces el par $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ no es, en general, un bialgebroide de Lie (ver [111]). Por tanto, para una variedad de Jacobi M, parece razonable considerar el par de algebroides de Jacobi $((A = TM \times \mathbb{R}, \phi_0 = (0, 1)),(A^* = T^*M \times \mathbb{R}, X_0 = (-E, 0)))$ en lugar del par de algebroides de Lie $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$. De hecho, probamos que los algebroides de Jacobi (A, ϕ_0) y (A^*, X_0) satisfacen algunas condiciones de compatibilidad. Estos resultados nos sugieren introducir, en la Sección 3.3, la definición de un bialgebroide de Jacobi como un par de algebroides de Jacobi en dualidad que son compatibles en un cierto sentido. Si M es una variedad de Jacobi entonces el par $((A = TM \times \mathbb{R}, \phi_0 = (0, 1)),(A^* = T^*M \times \mathbb{R}, X_0 = (-E, 0)))$ es un bialgebroide de Jacobi. Recíprocamente, probamos que una estructura de Jacobi puede ser definida en el espacio base de un bialgebroide de Jacobi. Después de esto, mostramos una caracterización interesante de bialgebroides de Jacobi que fue probada por Grabowski and Marmo en [33], a saber, si (A, ϕ_0) y (A^*, X_0) son un par de algebroides de Jacobi en dualidad entonces $((A, \phi_0), (A^*, X_0))$ es un bialgebroide de Jacobi si y sólo si la X_0-diferencial de A^* es una derivación con respecto a $(\oplus_k \Gamma(\wedge^k A), [\![\, , \,]\!]'^{\phi_0})$, donde $\oplus_k \Gamma(\wedge^k A)$ es el espacio de las multi-secciones de A y $[\![\, , \,]\!]'^{\phi_0}$ es el ϕ_0-corchete de Schouten modificado, que está definido por

$$[\![P, Q]\!]'^{\phi_0} = (-1)^{p+1} [\![P, Q]\!]^{\phi_0},$$

para $P \in \Gamma(\wedge^p A)$ y $Q \in \Gamma(\wedge^* A)$.

Si $((A, \phi_0), (A^*, X_0))$ es un bialgebroide de Jacobi y los 1-cociclos ϕ_0 y X_0 se anulan entonces el pair (A, A^*) es un bialgebroide de Lie. Este y otros ejemplos interesantes, tales como los bialgebroides de Jacobi triangulares (que generalizan los bialgebroides de Lie triangulares [83]) y el bialgebroide de Jacobi asociado con una estructura de Poisson exacta, son descritos en la Sección 3.4, mostrando en cada caso cuál es la estructura de Jacobi inducida en el espacio base M.

Es bien conocido que el producto de una variedad de Jacobi con $\mathbb{R}$, dotado con la Poissonización de la estructura de Jacobi, es una variedad de Poisson (ver [74] y Sección 1.1.6). En la Sección 3.5 de este Capítulo, mostramos un resultado similar para bialgebroides de Jacobi. Más precisamente, probamos que si $((A, \phi_0), (A^*, X_0))$ es un bialgebroide de Jacobi sobre M entonces es posible definir una estructura de bialgebroide de Lie en el par de fibrados vectoriales duales $(A \times \mathbb{R}, A^* \times \mathbb{R})$ sobre $M \times \mathbb{R}$, de tal manera que la estructura de Poisson inducida en $M \times \mathbb{R}$ es justamente la Poissonización de la estructura de Jacobi en M (ver Teorema 3.29). Usando este resultado, mostramos que los bialgebroides de Jacobi satisfacen un teorema de dualidad, esto es, si $((A, \phi_0), (A^*, X_0))$ es un bialgebroide de Jacobi, también lo es $((A^*, X_0), (A, \phi_0))$.

Finalmente, en la última sección del Capítulo 3 (Sección 3.6), obtenemos una caracterización de bialgebroides de Jacobi en términos de morfismos de algebroides de Jacobi (ver Teorema 3.34). Como consecuencia, deducimos que los bialgebroides de Lie puede ser caracterizados en términos de morfismos de algebroides de Lie. Esta última caracterización fue probada por Mackenzie y Xu en [83].

CAPITULO 4

El objetivo del Capítulo 4 es estudiar las biálgebras de Jacobi, esto es, los bialgebroides de Jacobi para los que el espacio base es un punto. Comenzamos el Capítulo, en la Seccion 4.1, estudiando varios aspectos relacionados con estructuras de Jacobi algebraicas. Estas últimas pueden ser consideradas como una versión sobre un álgebra de Lie de las estructuras de Jacobi sobre una variedad. Entre los ejemplos de estructuras de Jacobi algebraicas encontramos las álgebras de Lie-localmente conforme simplécticas (una generalización de las álgebras de Lie simplécticas [76]) y las álgebras de Lie-contacto. Para este último tipo de estructuras, damos una demostración directa de un resultado que fue probado por Diatta [25]. De hecho, mostramos que si $\mathfrak{g}$ es un álgebra de Lie compacta dotada con una estructura de contacto algebraica, entonces $\mathfrak{g}$ es isomorfa al álgebra de Lie $\mathfrak{su}(2)$ del grupo unitario especial $SU(2)$. Además, describimos todas las estructuras de contacto algebraicas en $\mathfrak{su}(2)$.

En la Sección 4.2, mostramos que la X_0-diferencial $d_*^{X_0}$ de $\mathfrak{g}^*$ en una biálgebra de Jacobi $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ es un 1-cociclo con respecto a cierta representación del álgebra de Lie $\mathfrak{g}$ sobre $\wedge^2\mathfrak{g}$. Motivados por este hecho, proponemos un método para obtener biálgebras de Jacobi donde $d_*^{X_0}$ es un 1-coborde (ver Teorema 4.8). Este método es una generalización del método de la ecuación de Yang-Baxter para biálgebras de Lie y, además, nos permite obtener biálgebras de Jacobi desde estructuras de Jacobi algebraicas. Para ilustrar la teoría, presentamos algunos ejemplos de biálgebras de Jacobi en la Sección 4.3.

Varios autores han dedicado especial atención al estudio de las biálgebras de Lie compactas. Un importante resultado en esta dirección es el siguiente

[81] (ver también [86]): todo grupo de Lie semisimple compacto y conexo admite una estructura de grupo de Lie-Poisson no trivial. En la Sección 4.4, describimos la estructura de una biálgebra de Jacobi $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, donde $\mathfrak{g}$ es un álgebra de Lie compacta y $\phi_0 \neq 0$ ó $X_0 \neq 0$ (ver Teoremas 4.18 y 4.20). En particular, deducimos que, aparte del álgebra de Lie abeliana de dimensión par, el único ejemplo no trivial de biálgebra de Lie compacta $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ se obtiene cuando $\mathfrak{g}$ es el álgebra de Lie $\mathfrak{u}(2)$ del grupo unitario $U(2)$.

CAPITULO 5

Finalizamos la Memoria introduciendo, en el Capítulo 5, los grupoides de Jacobi como una generalización de los grupoides de Poisson y de contacto y de tal manera que los bialgebroides de Jacobi son los invariantes infinitesimales de los grupoides de Jacobi. Como en el caso de los grupoides de contacto, comenzamos con un grupoide de Lie $G \rightrightarrows M$, una estructura de Jacobi (Λ, E) en G y una función multiplicativa $\sigma : G \to \mathbb{R}$. Entonces, como en el caso de los grupoides de Poisson, consideramos el morfismo de fibrados vectoriales $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ inducido por la estructura de Jacobi (Λ, E). La función multiplicativa σ induce, de manera natural, una acción del grupoide tangente $TG \rightrightarrows TM$ sobre la proyección canónica $\pi_1 : TM \times \mathbb{R} \to TM$ obteniéndose un grupoide acción $TG \times \mathbb{R}$ sobre $TM \times \mathbb{R}$. Así, es necesario introducir una estructura de grupoide de Lie adecuada en $T^*G \times \mathbb{R}$ sobre A^*G. De hecho, probamos que si AG es el algebroide de Lie de un grupoide de Lie arbitrario $G \rightrightarrows M$, $\sigma : G \to \mathbb{R}$ es una función multiplicativa, $\bar{\pi}_G : T^*G \times \mathbb{R} \to G$ es la proyección canónica y η_G es la 1-forma de contacto canónica en $T^*G \times \mathbb{R}$ entonces,$(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ \bar{\pi}_G)$ es un grupoide de contacto de tal manera que la estructura de Jacobi en A^*G es justamente la estructura de Jacobi homogénea $(\Lambda_{(A^*G,\phi_0)}, E_{(A^*G,\phi_0)})$ inducida por el algebroide de Lie y el 1-cociclo ϕ_0 que viene de la función multiplicativa σ (ver Teoremas 5.7 y 5.10).

Motivados por los resultados anteriores, en Sección 5.2, introducimos la definición de un grupoide de Jacobi como sigue. Sea $G \rightrightarrows M$ un grupoide de Lie, (Λ, E) una estructura de Jacobi en G y $\sigma : G \to \mathbb{R}$ una función multiplicativa. Entonces, $(G \rightrightarrows M, \Lambda, E, \sigma)$ es un grupoide de Jacobi si la aplicación $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ es un morfismo de grupoides de Lie sobre alguna aplicación $\varphi_0 : A^*G \to TM \times \mathbb{R}$. En esta Sección, también obtenemos las propiedades principales de este tipo de grupoides. Por otra parte, los grupoides de Poisson y de contacto son grupoides de Jacobi. Estos y otros ejemplos interesantes, como los grupoides localmente conforme simplécticos o los grupoides de Jacobi para los que el espacio base es un punto (llamados grupos de Lie-Jacobi), son tratados en la Sección 5.3.

En la última sección del Capítulo 5 (Sección 5.4), probamos que los bialgebroides de Jacobi son los invariantes infinitesimales de los grupoides de Jacobi. Para este propósito, procedemos como sigue. Si $(G \rightrightarrows M, \Lambda, E, \sigma)$ es un grupoide de Jacobi entonces, usando algunos resultados sobre subvariedades coisótropas de una variedad de Jacobi obtenidos en la Sección 5.4.1, mostramos que el fibrado vectorial A^*G admite una estructura de algebroide de Lie y la función multiplicativa σ (respectivamente, el campo de vectores E) induce un 1-cociclo ϕ_0 (respectivamente, X_0) en AG (respectivamente, A^*G) de tal manera que $((AG, \phi_0), (A^*G, X_0))$ es un bialgebroide de Jacobi (ver Teorema 5.25). Varios ejemplos ilustran este resultado.

También probamos un recíproco al resultado anterior. Para ello nos situamos en las siguientes hipótesis: Sea $((AG, \phi_0), (A^*G, X_0))$ un bialgebroide de Jacobi donde AG es el algebroide de Lie de un grupoide de Lie $G \rightrightarrows M$ α-conexo y α-simplemente conexo. Entonces, existe una única función multiplicativa $\sigma : G \to \mathbb{R}$ y una única estructura de Jacobi (Λ, E) en G que hace de $(G \rightrightarrows M, \Lambda, E, \sigma)$ un grupoide de Jacobi con bialgebroide de Jacobi $((AG, \phi_0), (A^*G, X_0))$ (ver Teorema 5.33). Los dos resultados previos generalizan los obtenidos por Mackenzie and Xu [83, 85] para grupoides de Poisson. Como otra aplicación, mostramos que dada una variedad de Jacobi (M, Λ_0, E_0) siempre existe, al menos localmente, un grupoide de contacto $(G \rightrightarrows M, \eta, \sigma)$ tal que AG es isomorfo al algebroide de Lie $T^*M \times \mathbb{R} \to M$. Este resultado fue probado por primera vez en [23] (ver también [2]). Finalmente, en el caso particular de un bialgebroide de Jacobi triangular integrable, damos una expresión explícita de la estructura de Jacobi (Λ, E) en el correspondiente grupoide de Jacobi.

Department of Fundamental Mathematics
Section of Geometry and Topology

LIE GROUPS AND GROUPOIDS AND JACOBI STRUCTURES

David Iglesias Ponte

A Memory fulfilled under the supervision of Prof. Juan Carlos Marrero González to opt for the degree of Doctor by the University of La Laguna

La Laguna, April 2003

Contents

Notation

All the manifolds which are going to be considered in this Memory will be connected. Moreover, if M is a smooth manifold, we will use the following notation:

- $C^\infty(M, \mathbb{R})$ is the algebra of C^∞ real-valued functions on M.
- $\mathfrak{X}(M)$ is the Lie algebra of vector fields on M.
- $\Omega^k(M)$ is the space of k-forms on M.
- $\mathcal{V}^k(M)$ is the space of k-vector fields on M.
- $[\, ,\,]$ is the Schouten-Nijenhuis bracket on $\mathcal{V}^*(M) = \oplus_k \mathcal{V}^k(M)$.
- d_0 is the exterior differential on $\Omega^*(M) = \oplus_k \Omega^k(M)$.
- $\mathcal{L}_0$ is the usual Lie derivative.

Introduction

The notion of a Poisson manifold was introduced by Lichnerowicz in [73] (see also [115]). A Poisson structure on a manifold M is a Lie algebra bracket on the space of C^∞ real-valued functions on M such that it is a derivation on each argument with respect to the usual product of functions. One of the main motivations for the introduction of this notion is that Poisson manifolds play an important role in Classical Mechanics. In fact, Poisson brackets appear in a natural way in the study of some mechanical systems, particularly systems with constraints or in the reduction of systems with symmetry groups. But Poisson geometry is also relevant to the algebras of observables in quantum mechanics, as well as to more general noncommutative algebras. In fact, Kontsevich [59] has just shown that the classification of formal deformations of the algebra $C^\infty(M,\mathbb{R})$ for any manifold M is equivalent to the classification of formal families of Poisson structures on M.

Geometrically, the Poisson bracket induces a 2-vector Π on M, characterized by the relation

$$\{f,g\} = \Pi(d_0 f, d_0 g), \ \text{for}\ f,g \in C^\infty(M,\mathbb{R}).$$

Thus, the Jacobi identity for $\{\,,\}$ can be reinterpreted as the condition $[\Pi,\Pi]=0$ (see [3, 73, 110]).

Two interesting examples of Poisson manifolds are symplectic manifolds and Lie-Poisson structures on the dual of a Lie algebra [115]. In fact, a Poisson manifold is made with symplectic pieces in the sense that it admits a generalized foliation, the symplectic foliation, whose leaves are symplectic manifolds.

Another category with close relations to Poisson geometry is that of Lie algebroids. A Lie algebroid over a manifold M is a vector bundle A over M such that its space of sections $\Gamma(A)$ admits a Lie algebra bracket $[\![\,,]\!]$ and, moreover, there exists a bundle map ρ from A to TM, the anchor map, such that the corresponding homomorphism of $C^\infty(M,\mathbb{R})$-modules, also denoted by $\rho:\Gamma(A)\to\mathfrak{X}(M)$, satisfies a Leibniz relation, that is,

$$[\![X,fY]\!]=\rho(X)(f)Y+f[\![X,Y]\!],$$

for $X,Y\in\Gamma(A)$ and $f\in C^\infty(M,\mathbb{R})$ (see [82, 99]). Lie algebroids are a natural generalization of tangent bundles and real Lie algebras of finite dimension. But, there are many other interesting examples. For instance, the Lie algebra structure on the space of C^∞ real-valued functions on a Poisson manifold M allows us to define a Lie bracket on the space of 1-forms which endows the cotangent bundle T^*M with a natural Lie algebroid structure ([3, 14, 30, 65, 110]). There is also a connection in the reverse direction between Poisson manifolds and Lie algebroids: there is a one-to-one correspondence between Lie algebroid structures on a vector bundle $\tau:A\to M$ and homogeneous Poisson structures on the dual bundle A^* (see [14, 15]). In the particular case when M is a single point, that is, A is a real Lie algebra of finite dimension, the corresponding homogeneous Poisson structure on A^* is just the usual Lie-Poisson structure.

Two important operators associated with any Lie algebroid are the Schouten bracket and the differential (see [82]). Moreover, certain definitions and constructions related with graded Lie algebras, lifts of tensor fields over a manifold and Poisson structures may be generalized to arbitrary Lie algebroids (see [36, 37]). In the same direction, in [38] the authors introduced the notion of an algebroid as an extension of the definition of a Lie algebroid and they showed that many objects of the differential calculus on a manifold (associated with the canonical Lie algebroid structure on TM) may be obtained in the framework of a general algebroid.

From the Physics point of view, Lie algebroids can be used in order to give geometric descriptions of Lagrangian and Hamiltonian Mechanics. Thus, in [89], Martínez developed a geometric description of time-independent Lagrangian Mechanics on Lie algebroids in a parallel way to the usual formalism of Lagrangian Mechanics on the tangent bundle of a manifold. Other papers which study, in particular, aspects of time-independent Lagrangian systems on Lie algebroids are [8, 9, 13, 71, 119]. More recently, other authors (see [31, 91, 103]) have started an investigation on the possible generalization of the concept of a Lie algebroid to affine bundles. The main motivation was to create a geometrical model which would be a natural environment for a time-dependent version of Lagrange equations on Lie algebroids.

On the other hand, Lie algebroids may be considered as the infinitesimal invariants of Lie groupoids. To be precise, we recall first that a small category G over a base M is a set G equipped with source and target maps α and β from G to M, a unit section $\epsilon : M \to G$, and a multiplication operation $(g, h) \mapsto gh$ defined on the set $G^{(2)} = \{(g, h) \in G \times G \,/\, \alpha(g) = \beta(h)\}$ of composable pairs. These operations satisfy the conditions that $\alpha(gh) = \alpha(h)$, $\beta(gh) = \beta(g)$, $(gh)k = g(hk)$ when either side is defined, and the elements of $\epsilon(M)$ in G act as identities for multiplication. If all the elements of G have inverses with respect to these identities, G is called a groupoid. If G and M are manifolds, and the structural maps are smooth (one requires α and β to be submersions to insure that the domain of the multiplication is a manifold), then G is called a Lie groupoid (see, for instance, [82]).

The Lie algebra of vector fields on a Lie groupoid G contains a distinguished subalgebra $\mathfrak{X}_L(G)$ of fields which are left-invariant in a certain sense; these are sections of a vector bundle AG which can be identified with the normal bundle to $\epsilon(M)$ in G, and then with the kernel of $T\beta$. $T\alpha$ is then an anchor map $AG \to TM$ for a Lie algebroid on AG. We call this the Lie algebroid of the Lie groupoid G (see [82]). This construction generalizes the way to obtain the Lie algebra associated with any Lie group. Moreover, we know that any Lie algebra can be integrated to a connected and simply-connected Lie group. However, this is no longer true for Lie algebroids and Lie groupoids: Not all Lie algebroids can be integrated to Lie groupoids (see [82]). Recently, Crainic and Fernandes [18] have given the precise obstructions to integrate an arbitrary Lie algebroid to a Lie groupoid.

We have seen that the cotangent bundle T^*M of a Poisson manifold has a natural Lie algebroid structure derived from the Poisson bracket of functions. If there is a Lie groupoid G whose Lie algebroid is isomorphic to T^*M, we say that M is an integrable Poisson manifold. In this case, there exists a symplectic structure Ω on G for which the graph of the multiplication $\{(g,h,gh) \in G \times G \times G \,/\, \alpha(g) = \beta(h)\}$ is a lagrangian submanifold of the symplectic product $(G \times G \times G, \Omega \oplus \Omega \oplus -\Omega)$. A Lie groupoid $G \rightrightarrows M$ endowed with a symplectic structure satisfying this property is called a symplectic groupoid (see [14]). The base space of a symplectic groupoid is always a Poisson manifold. A canonical example of a symplectic groupoid is the cotangent bundle T^*G of an arbitrary Lie groupoid $G \rightrightarrows M$, where the symplectic structure is just the canonical symplectic structure Ω_{T^*G}. In this case, the base space is A^*G and the Poisson structure on A^*G is just the linear Poisson structure induced by the Lie algebroid AG. An interesting property is that the source map of a symplectic groupoid $\alpha : G \to M$ is a Poisson morphism. Global realizations for arbitrary Poisson manifolds were first found by Karasaev [55] and Weinstein [116] (recent results can be found in [10, 19]).

As we have just said, symplectic groupoids appeared in the 1980's with the independent work of Karasaev [55] and Weinstein [116] (see also the papers by Zakrzewski [122]), motivated by quantization problems. Meanwhile, a theory of Poisson-Lie groups had been developing through the work of Drinfeld [27] and Semenov-Tian-Shansky [104, 105] on completely integrable systems and quantum groups (see also [28, 63, 80, 81]). We remark that a connected simply connected abelian Poisson-Lie group is isomorphic to the dual space of a real Lie algebra endowed with the Lie-Poisson structure. It was therefore natural to unify the theory of Poisson-Lie groups and the theory of symplectic groupoids. For this purpose, Weinstein [117] introduced the notion of Poisson groupoid. A Poisson groupoid is a Lie groupoid $G \rightrightarrows M$ with a Poisson structure Π for which the graph of the partial multiplication is a coisotropic submanifold in the Poisson manifold $(G \times G \times G, \Pi \oplus \Pi \oplus -\Pi)$. If $(G \rightrightarrows M, \Pi)$ is a Poisson groupoid then there exists a Poisson structure on M such that the source projection $\alpha : G \to M$ is a Poisson morphism. Moreover, if AG is the Lie algebroid of G then the dual bundle A^*G to AG itself has a Lie algebroid structure. In addition, Poisson groupoids have interesting applications in the classical dynamical Yang-Baxter equation (see, for instance, [29, 68]).

In [83], Mackenzie and Xu proved that a Lie groupoid $G \rightrightarrows M$ endowed with a Poisson structure Π is a Poisson groupoid if and only if the bundle map $\#_\Pi : T^*G \to TG$ induced by Π is a morphism between the cotangent groupoid $T^*G \rightrightarrows A^*G$ and the tangent groupoid $TG \rightrightarrows TM$. This characterization was used in order to prove that Lie bialgebroids are the infinitesimal invariants of Poisson groupoids, that is, if $(G \rightrightarrows M, \Pi)$ is a Poisson groupoid then (AG, A^*G) is a Lie bialgebroid and, conversely, a Lie bialgebroid structure on the Lie algebroid of a (suitably simply connected) Lie groupoid can be integrated to a Poisson groupoid structure [79, 83, 85] (these results can be applied to obtain a new proof of a theorem of Karasaev [55] and Weinstein [116]). A Lie bialgebroid is a Lie algebroid A such that the dual vector bundle A^* also carries a Lie algebroid structure which is compatible in a certain way with that on A (see [83]). Lie bialgebroids generalize Drinfeld's Lie bialgebras [27]. Another important example of Lie bialgebroid is the associated one with a Poisson structure. More precisely, if M is a Poisson manifold with Poisson 2-vector Π and on TM (respectively, T^*M) we consider the trivial Lie algebroid structure (resp., the cotangent Lie algebroid structure associated with Π) then the pair (TM, T^*M) is a Lie bialgebroid.

The compatibility condition of a Lie bialgebroid have been expressed in the language of derivations of Gerstenhaber algebras in [61].

Although symplectic and Lie-Poisson structures are Poisson, there are interesting structures for Classical Mechanics, such as contact structures, which are not Poisson. An interesting generalization of Poisson manifolds, as well as of contact manifolds, are Jacobi manifolds. A Jacobi structure on a manifold M is a 2-vector Λ and a vector field E on M such that $[\Lambda, \Lambda] = 2E \wedge \Lambda$ and $[E, \Lambda] = 0$ [74]. If (M, Λ, E) is a Jacobi manifold one can define a bracket of functions, the Jacobi bracket, in such a way that the space $C^\infty(M, \mathbb{R})$ endowed with the Jacobi bracket is a local Lie algebra in the sense of Kirillov [57]. Conversely, a local Lie algebra structure on $C^\infty(M, \mathbb{R})$ induces a Jacobi structure on M [39, 57]. Some interesting examples of Jacobi manifolds, apart from Poisson and contact manifolds, are locally conformal symplectic (l.c.s.) manifolds. In fact, a Jacobi manifold admits a generalized foliation, called the characteristic foliation, whose leaves are contact or l.c.s. manifolds (see [24, 39, 57]).

There is also a relation between Jacobi manifolds and Lie algebroids. In fact, if M is an arbitrary manifold then the vector bundle $TM \times \mathbb{R} \to M$ possesses a natural Lie algebroid structure. Moreover, if M is a Jacobi manifold then the 1-jet bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure [56] (for a Jacobi manifold the vector bundle T^*M is not, in general, a Lie algebroid). However, the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not, in general, a Lie bialgebroid (see [111]).

The existence of a Lie algebroid structure associated with any Jacobi structure justifies the introduction of the notion of a contact groupoid. A contact groupoid $(G \rightrightarrows M, \eta, \sigma)$ is a Lie groupoid $G \rightrightarrows M$ endowed with a contact 1-form $\eta \in \Omega^1(G)$ and a function $\sigma \in C^\infty(G, \mathbb{R})$ such that

$$\eta(gh)(X_g \oplus_{TG} Y_h) = \eta(g)(X_g) + e^{\sigma(g)}\eta(h)(Y_h),$$

for $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$, where $\oplus_{TG}$ is the partial multiplication in the tangent Lie groupoid $TG \rightrightarrows TM$ (see [22, 23, 56, 69, 70]). Contact groupoids can be considered as the odd-dimensional counterpart of symplectic groupoids and they have applications in the prequantization of Poisson manifolds and in the integration of local Lie algebras associated to rank one vector bundles (see [22, 23]). In this case, the function σ is multiplicative and the base space M carries an induced Jacobi structure in such a way that the pair (α, e^{σ}) is a conformal Jacobi morphism. Thus, we can consider the 1-jet Lie algebroid $T^*M \times \mathbb{R} \to M$. In fact, the Lie algebroid AG of G is isomorphic to $T^*M \times \mathbb{R} \to M$.

As it is indicated in the title, the main purpose of this Memory is to study the relation that there exists between the theory of Lie groupoids (and Lie algebroids) and Jacobi manifolds.

A general scheme of this Memory is the following one:

- Chapter 1: Jacobi structures, Lie algebroids and Lie groupoids

  This is an introductory chapter which contains some generalities about Jacobi structures, Lie algebroids and Lie groupoids, such as their definition, several examples and properties which are going to be useful in the sequel.

- Chapter 2: Jacobi algebroids, homogeneous Jacobi structures and its characteristic foliation

  A Jacobi algebroid structure on a vector bundle is a Lie algebroid structure plus a 1-cocycle on it. In this chapter, we consider homogeneous Jacobi structures on vector bundles. We obtain a characterization of this type of structures and its relation with Jacobi algebroid structures. We also discuss some examples and applications. Finally, we prove that the leaves of the characteristic foliation of a homogeneous Jacobi structure on a vector space are the orbits of an action of a Lie group on the vector space and we describe such an action.

- Chapter 3: Jacobi structures and Jacobi bialgebroids

  After developing a differential calculus for Jacobi algebroids, we introduce the notion of a Jacobi bialgebroid (a generalization of the notion of a Lie bialgebroid) in such a way that a Jacobi manifold has associated a canonical Jacobi bialgebroid. As a kind of converse, we prove that a Jacobi structure can be defined on the base space of a Jacobi bialgebroid. We also show that it is possible to construct a Lie bialgebroid from a Jacobi bialgebroid and, as a consequence, we deduce a duality theorem. The definition of a Jacobi bialgebroid is illustrated with several examples. In the last part of the chapter, we obtain a characterization of Jacobi bialgebroids in terms of Jacobi algebroid morphisms.

- Chapter 4: Jacobi bialgebras

  We study in this chapter Jacobi bialgebroids over a single point, that is, Jacobi bialgebras. We propose a method generalizing the Yang-Baxter equation method to obtain Jacobi bialgebras and give some examples of them. Finally, we discuss compact Jacobi bialgebras.

- Chapter 5: Jacobi groupoids and Jacobi bialgebroids

  We finish the Memory introducing Jacobi groupoids as a generalization of Poisson and contact groupoids. It is also proved that Jacobi bialgebroids are the infinitesimal invariants of Jacobi groupoids.

Next, we will precisely give the contents of every chapter included in this Memory.

We start this Memory with Chapter 1, where we recall some definitions and results about Jacobi structures, Lie algebroids and Lie groupoids, which will be useful in the sequel. First of all, we recall the definition of a Jacobi manifold. Several examples are considered, including Poisson manifolds, as well as other interesting examples of Jacobi manifolds which are not Poisson, such as contact or locally conformal symplectic manifolds (see Section 1.1.2). These last structures are specially important since, roughly speaking, every Jacobi manifold is made from contact and locally conformal symplectic pieces. More precisely, we have that the leaves of the characteristic foliation of a Jacobi manifold are contact or locally conformal symplectic manifolds (see Section 1.1.3).

Poisson structures are examples of Jacobi structures, but there exists another relation between Jacobi and Poisson structures. In fact, if M is a Jacobi manifold then the product manifold $M \times \mathbb{R}$ admits an exact Poisson structure which is called the Poissonization of M. In Section 1.1.6 of this Chapter, we give a description of this structure on $M \times \mathbb{R}$.

It is well known that if M is a Jacobi manifold then the vector bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure [56]. In the second part of this Chapter (Section 1.2), we recall the definition of a Lie algebroid structure on a vector bundle A over a manifold M and the definition of two important operators associated with any Lie algebroid: the Schouten bracket of two multi-sections of A and the differential of a multi-section of the dual bundle A^* to A. The differential is a cohomology operator and it induces the so-called Lie algebroid cohomology complex with trivial coefficients. Several examples of Lie algebroids are considered in Section 1.2.2, describing all the elements associated with each of them. In particular, we present the relation between Lie algebroid structures on a vector bundle $\tau : A \to M$ and Poisson structures on the dual bundle $\tau^* : A^* \to M$ which are homogeneous with respect to Δ_{A^*}, the Liouville vector field of A^*.

The global objects corresponding with Lie algebroids are Lie groupoids. In the last section of Chapter 1, we recall the definition of a Lie groupoid and of a morphism between Lie groupoids. As in the case of a Lie group, one may consider left-invariant multivector fields on a Lie groupoid. In particular, left-invariant vector fields are closed with respect to the Schouten-Nijenhuis bracket and can be identified with sections of a vector bundle $AG \to M$. These facts permit the construction of a Lie algebroid structure on AG. Some examples of Lie groupoids are considered in Section 1.3.2, describing the associated Lie algebroid in each case.

In Chapter 1, we have recalled the one-to-one correspondence between Lie algebroid structures on a vector bundle $\tau : A \to M$ and homogeneous Poisson structures on the dual bundle A^*. Moreover, we show the relation between the homogeneity of a Poisson structure and the behaviour of the Poisson bracket with respect to linear functions. In fact, a Poisson structure on a vector bundle is homogeneous if and only if the Poisson structure is linear, i.e., linear functions are closed with respect to the Poisson bracket. In Chapter 2 of this Memory we extend this relation to the Jacobi setting. More precisely, in Section 2.1, we describe homogeneous Jacobi structures on a vector bundle, that is, Jacobi structures (Λ, E) on a vector bundle $A \to M$ such that Λ and E are homogeneous with respect to the Liouville vector field of A. Moreover, as in the Poisson case, we explain the relation between this homogeneity and the behaviour of the Jacobi bracket $\{\, , \}_{(\Lambda,E)}$ between linear and constant functions. In particular, we prove that the vector field E is the vertical lift of a certain section of $A \to M$ and there exists a homogeneous Poisson structure Π_A such that $\Lambda = \Pi_A + E \wedge \Delta_A$, Δ_A being the Liouville vector field of A (see Theorem 2.3).

In Section 2.2, we show the relation between homogeneous Jacobi structures on A^* and Lie algebroid structures on A. In fact, if (Λ, E) is a homogeneous Jacobi structure on A^* then we obtain that it induces not only a Lie algebroid structure on A, but also a 1-cocycle $\phi_0 \in \Gamma(A^*)$ in the cohomology complex of A with trivial coefficients. The vector field E is, up to the sign, the vertical lift $\phi_0^{\mathbf{v}} \in \mathfrak{X}(A^*)$ of ϕ_0, that is, $E = -\phi_0^{\mathbf{v}}$. Motivated by this result, we introduce the notion of a Jacobi algebroid as a pair formed with a Lie algebroid structure and a 1-cocycle on it. After showing a converse of this result, that is, to obtain a homogeneous Jacobi structure on $A^* \to M$ from a Jacobi algebroid structure $((\llbracket\, , \rrbracket, \rho), \phi_0)$ on A, we present some examples and applications in Section 2.3. Two interesting examples are: i) for an arbitrary manifold M, the Lie algebroid $A = TM \times \mathbb{R}$ and the 1-cocycle $\phi_0 = (0, -1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A^*)$, we prove that the resultant homogeneous Jacobi structure on the vector bundle $T^*M \times \mathbb{R} \to M$ is just the canonical contact structure η_M; and ii) for a Jacobi manifold (M, Λ, E), the Lie algebroid $A^* = T^*M \times \mathbb{R}$ and the 1-cocycle $X_0 = (-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(A)$, we deduce that the corresponding homogeneous Jacobi structure $(\Lambda_{(TM\times\mathbb{R},X_0)}, E_{(TM\times\mathbb{R},X_0)})$ on the vector bundle $TM \times \mathbb{R} \to M$ is given by

$$\Lambda_{(TM\times\mathbb{R},X_0)} = \Lambda^c + \frac{\partial}{\partial t} \wedge E^c - t\Big(\Lambda^{\mathbf{v}} + \frac{\partial}{\partial t} \wedge E^{\mathbf{v}}\Big), \quad E_{(TM\times\mathbb{R},X_0)} = E^{\mathbf{v}},$$

where Λ^c and E^c (resp. $\Lambda^{\mathbf{v}}$ and $E^{\mathbf{v}}$) is the complete (resp. vertical) lift to TM of Λ and E, respectively. This Jacobi structure was first introduced in [43] and it is the Jacobi counterpart to the tangent Poisson structure first used in [102] (see also [15, 35]). As an application of our construction, we obtain a Lie algebroid structure $([\![\, , \,]\!]\hat{}^{\phi_0}, \hat{\rho}^{\phi_0})$ on the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ from a Jacobi algebroid structure $(([\![\, , \,]\!], \rho), \phi_0)$ on $A \to M$.

In the last Section of Chapter 2 (Section 2.4), we prove that the leaves of the characteristic foliation of a homogeneous Jacobi structure on the dual space $\mathfrak{g}^*$ of a real vector space $\mathfrak{g}$ are the orbits of a certain action of a connected simply connected Lie group $\tilde{G}$ on $\mathfrak{g}^*$ and we describe the Jacobi structure induced on each of the leaves. As a consequence, we deduce a well-known result: if $\Pi_{\mathfrak{g}^*}$ is a linear Poisson structure on $\mathfrak{g}^*$ then $\mathfrak{g}$ is a Lie algebra and the leaves of the symplectic foliation of $\Pi_{\mathfrak{g}^*}$ are the orbits of the coadjoint representation associated with a connected simply connected Lie group G with Lie algebra $\mathfrak{g}$.

Motivated by the results obtained in Chapter 2, we introduce, in Chapter 3, a differential calculus associated with any Jacobi algebroid. If $(A, ([\![\, , \,]\!], \rho), \phi_0)$ is a Jacobi algebroid over M then the usual representation of the Lie algebra $\Gamma(A)$ on the space $C^\infty(M, \mathbb{R})$ given by the anchor map ρ can be modified and a new representation is obtained. The resultant cohomology operator d^{ϕ_0} is called the ϕ_0-differential of A. The ϕ_0-differential of A allows us to define, in a natural way, the ϕ_0-Lie derivative by a section $X \in \Gamma(A)$, $\mathcal{L}_X^{\phi_0}$, as the commutator of d^{ϕ_0} and the contraction by X, that is, $\mathcal{L}_X^{\phi_0} = d^{\phi_0} \circ i_X + i_X \circ d^{\phi_0}$. On the other hand, imitating the definition of the Schouten bracket of two multilinear first-order differential operators on the space of C^∞ real-valued functions on a manifold N (see [3]), we introduce the ϕ_0-Schouten bracket of a k-section P and a k'-section P' as the $(k+k'-1)$-section given by

$$[\![P,P']\!]^{\phi_0} = [\![P,P']\!] + (-1)^{k+1}(k-1)P\wedge(i_{\phi_0}P') - (k'-1)(i_{\phi_0}P)\wedge P',$$

where $[\![\,,\,]\!]$ is the usual Schouten bracket of A. For these operators, we describe some of their properties.

On the other hand, if M is a Jacobi manifold then, as we know, the 1-jet bundle $T^*M\times\mathbb{R}\to M$ admits a Lie algebroid structure [56]. However, if on the vector bundle $TM\times\mathbb{R}\to M$ we consider the natural Lie algebroid structure then the pair $(TM\times\mathbb{R}, T^*M\times\mathbb{R})$ is not, in general, a Lie bialgebroid (see [111]). Therefore, for a Jacobi manifold M, it seems reasonable to consider the pair of Jacobi algebroids $((A=TM\times\mathbb{R},\phi_0=(0,1)),(A^*=T^*M\times\mathbb{R},X_0=(-E,0)))$ instead of the pair of Lie algebroids $(TM\times\mathbb{R},T^*M\times\mathbb{R})$. In fact, we prove that the Jacobi algebroids (A,ϕ_0) and (A^*,X_0) satisfy some compatibility conditions. These results suggest us to introduce, in Section 3.3, the definition of a Jacobi bialgebroid as a pair of Jacobi algebroids in duality which are compatible in a certain sense. If M is a Jacobi manifold then the pair $((A=TM\times\mathbb{R},\phi_0=(0,1)),(A^*=T^*M\times\mathbb{R},X_0=(-E,0)))$ is a Jacobi bialgebroid. As a kind of converse, we prove that a Jacobi structure can be defined on the base space of a Jacobi bialgebroid. After this, we show an interesting characterization of Jacobi bialgebroids which was proved by Grabowski and Marmo in [33], namely, if (A,ϕ_0) and (A^*,X_0) are a pair of Jacobi algebroids in duality then $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid if and only if the X_0-differential of A^* is a derivation with respect to $(\oplus_k\Gamma(\wedge^kA),[\![\,,\,]\!]'^{\phi_0})$, where $\oplus_k\Gamma(\wedge^kA)$ is the space of multi-sections of A and $[\![\,,\,]\!]'^{\phi_0}$ is the modified ϕ_0-Schouten bracket which is defined by

$$[\![P,Q]\!]'^{\phi_0} = (-1)^{p+1}[\![P,Q]\!]^{\phi_0},$$

for $P\in\Gamma(\wedge^pA)$ and $Q\in\Gamma(\wedge^*A)$.

If $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid and the 1-cocycles ϕ_0 and X_0 vanish then the pair (A,A^*) is a Lie bialgebroid. This and other interesting examples, such as triangular Jacobi bialgebroids (which generalize triangular Lie bialgebroids [83]) and the Jacobi bialgebroid associated with an exact Poisson structure, are described in Section 3.4, showing in each case which is the induced Jacobi structure on the base space M.

It is well-known that the product of a Jacobi manifold M with $\mathbb{R}$, endowed with the Poissonization of the Jacobi structure, is a Poisson manifold (see [74] and Section 1.1.6). We show a similar result for Jacobi bialgebroids. Namely, we prove that if $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid over M then it is possible to define a Lie bialgebroid structure on the dual pair of vector bundles $(A \times \mathbb{R}, A^* \times \mathbb{R})$ over $M \times \mathbb{R}$, in such a way that the induced Poisson structure on $M \times \mathbb{R}$ is just the Poissonization of the Jacobi structure on M (Theorem 3.29 in Section 3.5). Using this result, we deduce that the Jacobi bialgebroids satisfy a duality theorem, that is, if $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid, so is $((A^*, X_0), (A, \phi_0))$.

Finally, in the last section of Chapter 3 (Section 3.6), we obtain a characterization of Jacobi bialgebroids in terms of Jacobi algebroid morphisms (see Theorem 3.34). As a consequence, we deduce that Lie bialgebroids may be characterized in terms of Lie algebroid morphisms. This characterization was proved by Mackenzie and Xu [83].

The purpose of Chapter 4 is to study Jacobi bialgebras, that is, Jacobi bialgebroids over a single point. We start the Chapter, in Section 4.1, considering several aspects of algebraic Jacobi structures, an algebraic version of the concept of Jacobi structures. Among the examples of algebraic Jacobi structures we find locally conformal symplectic Lie algebras (a generalization of symplectic Lie algebras [76]) and contact Lie algebras. For this last type of structures, we give a direct proof of a result which was proved by Diatta [25]. In fact, we show that if $\mathfrak{g}$ is a compact Lie algebra endowed with an algebraic contact structure, then $\mathfrak{g}$ is isomorphic to the Lie algebra $\mathfrak{su}(2)$ of the special unitary group $SU(2)$. In addition, we describe all the algebraic contact structures on $\mathfrak{su}(2)$.

In Section 4.2, we deal with a Jacobi bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$. In particular, we have that the X_0-differential $d_*^{X_0}$ of $\mathfrak{g}^*$ is a 1-cocycle with respect to a certain representation of the Lie algebra $\mathfrak{g}$ on $\wedge^2\mathfrak{g}$. Motivated by this fact, we propose a method to obtain Jacobi bialgebras where $d_*^{X_0}$ is a 1-coboundary (see Theorem 4.8). This method is a generalization of the Yang-Baxter equation method for Lie bialgebras and, moreover, allows us to obtain Jacobi bialgebras from algebraic Jacobi structures. To illustrate the theory, we present some examples of Jacobi bialgebras.

Several authors have devoted special attention to the study of compact Lie bialgebras and an important result in this direction is the following one [81] (see also [86]): every connected compact semisimple Lie group has a nontrivial Poisson Lie group structure. In Section 4.4, we describe the structure of a Jacobi bialgebra $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$, $\mathfrak{g}$ being a compact Lie algebra and $\phi_0 \neq 0$ or $X_0 \neq 0$ (see Theorems 4.18 and 4.20). In particular, we deduce that, apart from the abelian Lie algebra of even dimension, the only nontrivial example of compact Jacobi bialgebra $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,X_0))$ is obtained when $\mathfrak{g}$ is the Lie algebra $\mathfrak{u}(2)$ of the unitary group $U(2)$.

We finish the Memory introducing, in Chapter 5, Jacobi groupoids as a generalization of Poisson and contact groupoids and, in such a way that Jacobi bialgebroids may be considered as the infinitesimal invariants of Jacobi groupoids. As in the case of contact groupoids, we start with a Lie groupoid $G \rightrightarrows M$, a Jacobi structure (Λ, E) on G and a multiplicative function $\sigma : G \to \mathbb{R}$. Then, as in the case of Poisson groupoids, we consider the vector bundle morphism $\#_{(\Lambda,E)} : T^*G\times\mathbb{R} \to TG\times\mathbb{R}$ induced by the Jacobi structure (Λ, E). The multiplicative function σ induces, in a natural way, an action of the tangent groupoid $TG \rightrightarrows TM$ over the canonical projection $\pi_1 : TM \times \mathbb{R} \to TM$ obtaining an action groupoid $TG \times \mathbb{R}$ over $TM \times \mathbb{R}$. Thus, it is necessary to introduce a suitable Lie groupoid structure in $T^*G\times\mathbb{R}$ over A^*G. In fact, we prove that if AG is the Lie algebroid of an arbitrary Lie groupoid $G \rightrightarrows M$, $\sigma : G \to \mathbb{R}$ is a multiplicative function, $\tilde{\pi}_G : T^*G \times \mathbb{R} \to G$ is the canonical projection and η_G is the canonical contact 1-form on $T^*G \times \mathbb{R}$ then $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma \circ \tilde{\pi}_G)$ is a contact groupoid in such a way that the Jacobi structure on A^*G is just the homogeneous Jacobi structure $(\Lambda_{(A^*G,\phi_0)}, E_{(A^*G,\phi_0)})$ induced by the Lie algebroid AG and the 1-cocycle ϕ_0 which comes from the multiplicative function σ (see Theorems 5.7 and 5.10).

Motivated by the above results, in Section 5.2, we introduce the definition of a Jacobi groupoid as follows. Let $G \rightrightarrows M$ be a Lie groupoid, (Λ, E) be a Jacobi structure on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid if the map $\#_{(\Lambda,E)} : T^*G\times\mathbb{R} \to TG\times\mathbb{R}$ is a Lie groupoid morphism over some map $\varphi_0 : A^*G \to TM \times \mathbb{R}$. In this Section, we also obtain the main properties of this kind of groupoids. On the other hand, Poisson and contact groupoids are Jacobi groupoids. These and other interesting examples, such as locally conformal symplectic groupoids or Jacobi groupoids over a single point (called Jacobi-Lie groups), are treated in Section 5.3.

In the last section of Chapter 5 (Section 5.4), we prove that Jacobi bialgebroids are the infinitesimal invariants of Jacobi groupoids. For this purpose, we proceed as follows. If $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid then, using some results about coisotropic submanifolds of a Jacobi manifold obtained in Section 5.4.1, we show that the vector bundle A^*G admits a Lie algebroid structure and the multiplicative function σ (respectively, the vector field E) induces a 1-cocycle ϕ_0 (respectively, X_0) on AG (respectively, A^*G) in such a way that $((AG, \phi_0), (A^*G, X_0))$ is a Jacobi bialgebroid (see Theorem 5.25). Several examples illustrate this result.

We also prove a converse of the above statement. Namely, let $((AG, \phi_0), (A^*G, X_0))$ be a Jacobi bialgebroid, where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$. Then, there is a unique multiplicative function $\sigma : G \to \mathbb{R}$ and a unique Jacobi structure (Λ, E) on G that makes $(G \rightrightarrows M, \Lambda, E, \sigma)$ into a Jacobi groupoid with Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$ (see Theorem 5.33). The two previous results generalize those obtained by Mackenzie and Xu [83, 85] for Poisson groupoids. As another application, we show that given a Jacobi manifold (M, Λ_0, E_0) there always exists, at least locally, a contact groupoid $(G \rightrightarrows M, \eta, \sigma)$ such that AG is isomorphic to the 1-jet Lie algebroid $T^*M \times \mathbb{R} \to M$. This result was first proved in [23] (see also [2]). On the other hand, in the particular case of an integrable triangular Jacobi bialgebroid, we give an explicit expression of the Jacobi structure (Λ, E) on the corresponding Jacobi groupoid.

We finish this Memory with the references that we have mentioned throughout it, as well as with some other references in where some of the results that we have obtained here are included or others which have relation with this Memory.

CHAPTER 1

Jacobi structures, Lie algebroids and Lie groupoids

1.1 Local Lie algebras and Jacobi manifolds. Examples

This first Section of Chapter 1 contains some generalities about Jacobi manifolds: definition, examples and the description of the characteristic foliation of a Jacobi manifold.

1.1.1 Local Lie algebras and Jacobi manifolds

A *Jacobi structure* on a manifold M is a pair (Λ, E), where Λ is a 2-vector and E is a vector field on M satisfying the following properties:

$$[\Lambda, \Lambda] = 2E \wedge \Lambda, \qquad [E, \Lambda] = 0. \tag{1.1}$$

A manifold M endowed with a Jacobi structure is called a *Jacobi manifold*. If (M, Λ, E) is a Jacobi manifold then a bracket of functions (the *Jacobi bracket*) is defined by

$$\{f, g\}_{(\Lambda,E)} = \Lambda(d_0 f, d_0 g) + fE(g) - gE(f), \text{ for } f, g \in C^\infty(M, \mathbb{R}). \tag{1.2}$$

This bracket is $\mathbb{R}$-bilinear and satisfies the following properties:

i) Skew-symmetry: $\{f,g\}_{(\Lambda,E)} = -\{g,f\}_{(\Lambda,E)}$, for all $f,g \in C^\infty(M,\mathbb{R})$.

ii) It is a first-order differential operator on each of its arguments with respect to the ordinary multiplication of functions, that is,

$$\{f_1 f_2, g\}_{(\Lambda,E)} = f_1\{f_2,g\}_{(\Lambda,E)} + f_2\{f_1,g\}_{(\Lambda,E)} - f_1 f_2\{1,g\}_{(\Lambda,E)}, \quad (1.3)$$

for $f_1, f_2, g \in C^\infty(M,\mathbb{R})$.

iii) Jacobi identity:

$$\{f,\{g,h\}_{(\Lambda,E)}\}_{(\Lambda,E)} + \{g,\{h,f\}_{(\Lambda,E)}\}_{(\Lambda,E)} + \{h,\{f,g\}_{(\Lambda,E)}\}_{(\Lambda,E)} = 0,$$

for $f,g,h \in C^\infty(M,\mathbb{R})$.

Property *ii)* can be replaced by the following relation between the supports of the functions:

ii') support$\{f,g\}_{(\Lambda,E)} \subseteq$ (support f) $\cap$ (support g), for $f,g \in C^\infty(M,\mathbb{R})$.

Properties *i)*, *ii)* y *iii)* guarantee that the Jacobi bracket defines a *local Lie algebra* structure in the sense of Kirillov [57] on the space $C^\infty(M,\mathbb{R})$. Conversely, a local Lie algebra structure on $C^\infty(M,\mathbb{R})$ defines a Jacobi structure on M (see [39, 57]).

If the vector field E identically vanishes then, from (1.2), $\{\ ,\ \}_{(\Lambda,0)} = \{\ ,\ \}_\Lambda$ is a derivation in each argument and, therefore, $\{\ ,\ \}_\Lambda$ defines a *Poisson bracket* on M. In this case, (1.1) reduces to $[\Lambda,\Lambda] = 0$ and (M,Λ) is a *Poisson manifold*. Jacobi and Poisson manifolds were introduced by Lichnerowicz [73, 74] (see also [3, 24, 39, 72, 110, 115]).

1.1.2 Examples of Jacobi manifolds

In this Section, we will present some examples of Jacobi manifolds.

1.- *Poisson manifolds.* A Poisson structure on a manifold M is a 2-vector Π on M such that

$$[\Pi, \Pi] = 0.$$

We have seen in the previous Section that Poisson manifolds is a particular example of Jacobi manifolds. Some particular examples of Poisson manifolds are the following ones.

1a.- Symplectic manifolds. A *symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a closed non-degenerate 2-form on M. We define a Poisson 2-vector Π on M by

$$\Pi(\mu, \nu) = \Omega(\flat^{-1}(\mu), \flat^{-1}(\nu)), \tag{1.4}$$

for $\mu, \nu \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X\Omega$ (see [73]).

Using the classical theorem of Darboux, around every point of M there exist canonical coordinates $(q^1, \dots, q^m, p_1, \dots, p_m)$ on M such that

$$\Omega = \sum_{i=1}^{m} d_0 q^i \wedge d_0 p_i, \qquad \Pi = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

1b.- Lie-Poisson structures. Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a real Lie algebra of dimension n with Lie bracket $[\ ,\ ]_{\mathfrak{g}}$ and denote by $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Given two functions $f, h \in C^\infty(\mathfrak{g}^*, \mathbb{R})$, we define $\{f, h\}$ as follows. For a point $x \in \mathfrak{g}^*$, we linearize f and h, namely, we take the differential of f and h at x, $(d_0 f)(x)$ and $(d_0 h)(x)$, and identify them with two elements $\hat{f}, \hat{h} \in \mathfrak{g}$. Thus, $[\hat{f}, \hat{h}]_{\mathfrak{g}} \in \mathfrak{g}$, and we define

$$\{f, h\}(x) = \langle x, [\hat{f}, \hat{h}]_{\mathfrak{g}} \rangle.$$

$\{\ ,\ \}$ is the so-called *Lie-Poisson bracket* on $\mathfrak{g}^*$ (see [110, 115]).

If $\Pi_{\mathfrak{g}^*}$ is the corresponding Poisson 2-vector on $\mathfrak{g}^*$ and (v_i) are global coordinates for $\mathfrak{g}^*$ obtained from a basis, we have that

$$\Pi_{\mathfrak{g}^*} = \sum_{i<j} \sum_{k} c_{ij}^k v_k \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j}, \tag{1.5}$$

c_{ij}^k being the structure constants of $\mathfrak{g}$ with respect to the basis.
From (1.5), it follows that

$$(\mathcal{L}_0)_{\Delta_{\mathfrak{g}^*}}\Pi_{\mathfrak{g}^*} = -\Pi_{\mathfrak{g}^*}, \tag{1.6}$$

where $\Delta_{\mathfrak{g}^*}$ is the radial vector field on $\mathfrak{g}^*$. Note that the expression of $\Delta_{\mathfrak{g}^*}$ with respect to the coordinates (v_i) is

$$\Delta_{\mathfrak{g}^*} = \sum_{i=1}^{n} v_i \frac{\partial}{\partial v_i}. \tag{1.7}$$

2.- *Contact manifolds.* Let M be a $(2m+1)$-dimensional manifold and η be a 1-form on M. We say that η is a contact 1-form if $\eta \wedge (d_0\eta)^m \neq 0$ at every point. In such a case, (M, η) is termed a *contact manifold* (see, for example, [4, 72, 74]). If (M, η) is a contact manifold, we define a 2-vector Λ and a vector field E on M as follows

$$\Lambda(\mu, \nu) = d_0\eta(\flat^{-1}(\mu), \flat^{-1}(\nu)), \quad E = \flat^{-1}(\eta), \tag{1.8}$$

for all $\mu, \nu \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M, \mathbb{R})$-modules given by $\flat(X) = i_X(d_0\eta) + \eta(X)\eta$. Then, (M, Λ, E) is a Jacobi manifold (see [74]). The vector field E is just the *Reeb vector field* of M and it is characterized by the relations

$$i_E\eta = 1, \quad i_E(d_0\eta) = 0. \tag{1.9}$$

Using the generalized Darboux theorem, we deduce that around every point of M there exist canonical coordinates $(t, q^1, \ldots, q^m, p_1, \ldots, p_m)$ such that (see [4, 72, 74])

$$\eta = d_0t - \sum_{i=1}^{m} p_i d_0 q^i, \qquad \Lambda = \sum_{i=1}^{m} \Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t} \Big) \wedge \frac{\partial}{\partial p_i}, \qquad E = \frac{\partial}{\partial t}. \tag{1.10}$$

3.- *Locally conformal symplectic manifolds.* An *almost symplectic manifold* is a pair (M, Ω), where M is an even-dimensional manifold and Ω is a non-degenerate 2-form on M. An almost symplectic manifold is said to be *locally* *conformal symplectic (l.c.s.)* if for each point $x \in M$ there is an open neighborhood U such that $d_0(e^{-f}\Omega) = 0$, for some function $f : U \longrightarrow \mathbb{R}$ (see, for example, [39, 57, 109]). So, $(U, e^{-f}\Omega)$ is a symplectic manifold. If $U = M$ then M is said to be a *globally conformal symplectic (g.c.s.)* manifold. An almost symplectic manifold (M, Ω) is l.(g.)c.s. if and only if there exists a closed (exact) 1-form ω such that

$$d_0\Omega = \omega \wedge \Omega. \tag{1.11}$$

The 1-form ω is called the *Lee 1-form* of M. It is obvious that l.c.s. manifolds with Lee 1-form identically zero are just symplectic manifolds.

In a similar way that for contact manifolds, we define a 2-vector Λ and a vector field E on M which are given by

$$\Lambda(\mu,\nu) = \Omega(\flat^{-1}(\mu), \flat^{-1}(\nu)), \qquad E = \flat^{-1}(\omega)\ , \tag{1.12}$$

for $\mu, \nu \in \Omega^1(M)$, where $\flat : \mathfrak{X}(M) \longrightarrow \Omega^1(M)$ is the isomorphism of $C^\infty(M,\mathbb{R})$-modules defined by $\flat(X) = i_X\Omega$. Then, (M, Λ, E) is a Jacobi manifold (see [39]).

Using the classical theorem of Darboux, around every point of M there exist canonical coordinates $(q^1, \ldots, q^m, p_1, \ldots, p_m)$ and a local differentiable function f such that

$$\Omega = e^f \sum_{i=1}^m d_0q^i \wedge d_0p_i, \qquad \omega = d_0f = \sum_{i=1}^m \Big(\frac{\partial f}{\partial q^i} d_0q^i + \frac{\partial f}{\partial p_i} d_0p_i\Big),$$
$$\Lambda = e^{-f} \sum_{i=1}^m \Big(\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}\Big), \qquad E = e^{-f} \sum_{i=1}^m \Big(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}\Big).$$

1.1.3 The characteristic foliation of a Jacobi manifold

Let (M, Λ, E) be a Jacobi manifold. Define a homomorphism of $C^\infty(M,\mathbb{R})$-modules $\#_\Lambda : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$ by

$$(\#_\Lambda(\mu))(\nu) = \Lambda(\mu,\nu), \tag{1.13}$$

for $\mu, \nu \in \Omega^1(M)$. This homomorphism can be extended to a homomorphism, which we also denote by $\#_\Lambda$, from the space $\Omega^k(M)$ to the space $\mathcal{V}^k(M)$ by putting:

$$\#_\Lambda(f) = f, \quad \#_\Lambda(\mu)(\mu_1, \ldots, \mu_k) = (-1)^k \mu(\#_\Lambda(\mu_1), \ldots, \#_\Lambda(\mu_k)), \tag{1.14}$$

for $f \in C^\infty(M, \mathbb{R})$, $\mu \in \Omega^k(M)$ and $\mu_1, \ldots, \mu_k \in \Omega^1(M)$.

Remark 1.1 *i)* If M is a contact manifold with Reeb vector field E, then $\#_\Lambda(\mu) = -\flat^{-1}(\mu) + \mu(E)E$, for all $\mu \in \Omega^1(M)$. In particular, $\eta(\#_\Lambda(\mu)) = 0$.

ii) If (M, Ω) is a l.c.s. manifold with Lee 1-form ω then $\#_\Lambda(\mu) = -\flat^{-1}(\mu)$, for all $\mu \in \Omega^1(M)$. In particular, $\#_\Lambda(\omega) = -E$.

If f is a C^∞-differentiable real-valued function on a Jacobi manifold M, the vector field $\mathcal{H}_f^{(\Lambda,E)}$ defined by

$$\mathcal{H}_f^{(\Lambda,E)} = \#_\Lambda(d_0 f) + fE \tag{1.15}$$

is called the *hamiltonian vector field* associated with f. It should be noticed that the hamiltonian vector field associated with the constant function 1 is just E. A direct computation proves that (see [74, 87])

$$[\mathcal{H}_f^{(\Lambda,E)}, \mathcal{H}_g^{(\Lambda,E)}] = \mathcal{H}^{(\Lambda,E)}_{\{f,g\}_{(\Lambda,E)}}, \tag{1.16}$$

which shows that the mapping $C^\infty(M, \mathbb{R}) \longrightarrow \mathfrak{X}(M)$, $f \mapsto \mathcal{H}_f^{(\Lambda,E)}$, is a homomorphism between the Lie algebras $(C^\infty(M, \mathbb{R}), \{\, ,\, \}_{(\Lambda,E)})$ and $(\mathfrak{X}(M), [\, ,\,])$.

Now, for every $x \in M$, we consider the subspace $\mathcal{F}_x^{(\Lambda,E)}$ of T_xM generated by all the hamiltonian vector fields evaluated at the point x. In other words, $\mathcal{F}_x^{(\Lambda,E)} = (\#_\Lambda)_x(T_x^*M) + \langle E_x \rangle$. Since $\mathcal{F}^{(\Lambda,E)}$ is involutive and finitely generated, one easily follows that $\mathcal{F}^{(\Lambda,E)}$ defines a generalized foliation in the sense of Sussmann [106], which is called the *characteristic foliation* (see [24, 39]). Moreover, the Jacobi structure of M induces a Jacobi structure on each leaf. In fact, if L_x is the leaf over a point x of M and $E_x \notin Im(\#_\Lambda)_x$ (or, equivalently, the dimension of L_x is odd) then L_x is a contact manifold with the induced Jacobi structure. More precisely, if y is a point of L_x and η^{L_x} is the contact structure on L_x then $T_y(L_x) = \{\mathcal{H}_f^{(\Lambda,E)}(y) \,/\, f \in C^\infty(M,\mathbb{R})\}$ and

$$\eta^{L_x}(y)(\mathcal{H}_f^{(\Lambda,E)}(y)) = f(y). \tag{1.17}$$

If $E_x \in Im(\#_\Lambda)_x$ (or, equivalently, the dimension of L_x is even) then L_x is a l.c.s. manifold and the l.c.s. structure $(\Omega^{L_x}, \omega^{L_x})$ on L_x is given by

$$\begin{aligned} \Omega^{L_x}(y)(\mathcal{H}_f^{(\Lambda,E)}(y), \mathcal{H}_g^{(\Lambda,E)}(y)) &= \Lambda(y)(d_0 f(y), d_0 g(y)) + f(y)E(y)(g) \\ &\quad - g(y)E(y)(f), \\ \omega^{L_x}(y)(\mathcal{H}_f^{(\Lambda,E)}(y)) &= -E(y)(f), \end{aligned} \tag{1.18}$$

for $y \in L_x$ and $f, g \in C^\infty(M,\mathbb{R})$ (for a detailed study of the characteristic foliation, we refer to [24, 39]). In the particular case when M is a Poisson manifold then, from (1.13) and (1.15), we deduce that the characteristic foliation of M is just the *canonical symplectic foliation* of M (see [110, 115]).

Remark 1.2 For a symplectic, contact or l.c.s. manifold M there exists a unique leaf of its characteristic foliation: the manifold M. Conversely, if M is a *transitive Jacobi manifold*, that is, if $\mathcal{F}_x^{(\Lambda,E)} = T_x M$, for all $x \in M$, then M is a contact or l.c.s. manifold (see [24, 39, 57]).

1.1.4 Conformal changes of Jacobi manifolds and conformal Jacobi morphisms

Let (M, Λ, E) be a Jacobi manifold and $a \in C^\infty(M,\mathbb{R})$ be a positive function. Let us consider the 2-vector Λ_a and the vector field E_a on M given by

$$\Lambda_a = a\Lambda, \qquad E_a = \mathcal{H}_a^{(\Lambda,E)} = \#_\Lambda(d_0 a) + a\,E.$$

Then, the pair (Λ_a, E_a) is a Jacobi structure on M. The Jacobi brackets $\{\,,\,\}_{(\Lambda,E)}$ and $\{\,,\,\}_{(\Lambda_a,E_a)}$ are related by

$$\{f, g\}_{(\Lambda_a,E_a)} = \frac{1}{a}\{a\,f, a\,g\}_{(\Lambda,E)},$$

for $f, g \in C^\infty(M,\mathbb{R})$.

In this case, we say that the Jacobi structures (Λ, E) and (Λ_a, E_a) are *conformally equivalent* (see [24, 39, 74]).

Let $\psi : (M_1, \Lambda_1, E_1) \to (M_2, \Lambda_2, E_2)$ be a differentiable mapping between the Jacobi manifolds (M_1, Λ_1, E_1) and (M_2, Λ_2, E_2). Suppose that $\{\, ,\, \}_{(\Lambda_1, E_1)}$ (respectively, $\{\, ,\, \}_{(\Lambda_2, E_2)}$) is the Jacobi bracket on M_1 (respectively, M_2).

The mapping ψ is said to be a *Jacobi morphism* if

$$\{f_2, g_2\}_{(\Lambda_2, E_2)} \circ \psi = \{f_2 \circ \psi, g_2 \circ \psi\}_{(\Lambda_1, E_1)},$$

for $f_2, g_2 \in C^\infty(M_2, \mathbb{R})$. Equivalently, ψ is a Jacobi morphism if

$$\Lambda_1(\psi^*\mu, \psi^*\nu) = \Lambda_2(\mu, \nu) \circ \psi, \qquad \psi_* E_1 = E_2,$$

for $\mu, \nu \in \Omega^1(M_2)$.

Now, if a is a positive function on M_1 the pair (ψ, a) is called a *conformal Jacobi morphism* if the mapping ψ is a Jacobi morphism between the Jacobi manifolds $(M_1, (\Lambda_1)_a, (E_1)_a)$ and (M_2, Λ_2, E_2). The conformal Jacobi isomorphisms are the conformal Jacobi morphisms (ψ, a) such that ψ is a diffeomorphism (see [24]).

1.1.5 Coisotropic submanifolds

In this Section, we will give a definition which will be useful in the following.

Definition 1.3 *Let S be a submanifold of a manifold M and Λ be an arbitrary 2-vector. S is said to be* coisotropic *(with respect to Λ) if*

$$\#_\Lambda((T_xS)^\circ) \subseteq T_xS,$$

for $x \in S$, $(T_xS)^\circ$ being the annihilator space of T_xS.

Remark 1.4 If Π (respectively, (Λ, E)) is a Poisson structure (respectively, a Jacobi structure) on M then we recover the notion of a coisotropic submanifold of the Poisson manifold (M, Π) [72, 117] (respectively, coisotropic submanifold of a Jacobi manifold (M, Λ, E) [43]) .

1.1.6 The Poissonization of a Jacobi manifold

Let (Λ, E) be a 2-vector and a vector field on a manifold M. Then, we can consider the 2-vector Π on $M \times \mathbb{R}$ given by

$$\Pi = e^{-t}\Big(\Lambda + \frac{\partial}{\partial t} \wedge E\Big), \tag{1.19}$$

where t is the usual coordinate on $\mathbb{R}$. The 2-vector Π is homogeneous with respect to the vector field $\frac{\partial}{\partial t}$, that is,

$$[\frac{\partial}{\partial t}, \Pi] = -\Pi.$$

In fact, if Π is a 2-vector on $M \times \mathbb{R}$ such that $[\frac{\partial}{\partial t}, \Pi] = -\Pi$ then there exists a 2-vector Λ and a vector field E on M such that Π is given by (1.19). Moreover, (Λ, E) is a Jacobi structure on M if and only if Π defines a Poisson structure on $M \times \mathbb{R}$ (see [74]). The manifold $M \times \mathbb{R}$ endowed with the structure Π is called the *Poissonization of the Jacobi manifold* (M, Λ, E).

Examples 1.5 **1.-** *The Poissonization of a Poisson manifold.*
Let (M, Λ) be a Poisson manifold. We have seen that it can be considered as a Jacobi manifold, where $E = 0$. In this case, the Poissonization of (M, Λ) is again a Poisson structure $\Pi = e^{-t}\Lambda$.

2.- *The Poissonization of a contact structure.*
Let η be a 1-form on a manifold M of dimension $2m + 1$. If we consider on the product manifold $M \times \mathbb{R}$, the 2-form Ω given $\Omega = e^t\Big(d_0\eta + d_0 t \wedge \eta\Big)$, it follows that η is a contact 1-form on M if and only if Ω is a symplectic 2-form on $M \times \mathbb{R}$ (see, for instance, [72]).

We denote by (Λ, E) the Jacobi structure associated with η and by Π the Poisson structure coming from the symplectic form Ω. A direct computation, using (1.4) and (1.8), shows that $\Pi = e^{-t}\Big(\Lambda + \frac{\partial}{\partial t} \wedge E\Big)$. Thus, the Poissonization of a contact manifold (M, η) is the symplectic manifold $(M \times \mathbb{R}, \Omega)$.

1.2 Lie algebroids. Examples

A category with close relations to Poisson and Jacobi geometry is that of Lie algebroids. In this Section, we will recall the definition of a Lie algebroid and of the differential calculus associated to them. Moreover, we illustrate the theory with several examples.

1.2.1 Lie algebroids

A *Lie algebroid* A over a manifold M is a vector bundle A over M together with a Lie bracket $[\![\, , \,]\!]$ on the space $\Gamma(A)$ of the global cross sections of $A \to M$ and a bundle map $\rho\colon A \to TM$, called the *anchor map*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M,\mathbb{R})$-modules induced by the anchor map then

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y,$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M,\mathbb{R})$. The triple $(A, [\![\, , \,]\!], \rho)$ is called a *Lie algebroid over* M (see [82, 99]).

Remark 1.6 If $(A, [\![\, , \,]\!], \rho)$ is a Lie algebroid over M then the anchor map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(A), [\![\, , \,]\!])$ and $(\mathfrak{X}(M), [\, , \,])$.

If $(A, [\![\, , \,]\!], \rho)$ is a Lie algebroid, the Lie bracket on $\Gamma(A)$ can be extended to the so-called *Schouten bracket* $[\![\, , \,]\!]$ on the space $\Gamma(\wedge^* A) = \oplus_k \Gamma(\wedge^k A)$ of multi-sections of A in such a way that $(\oplus_k \Gamma(\wedge^k A), \wedge, [\![\, , \,]\!])$ is a graded Lie algebra. In fact, the Schouten bracket satisfies the following properties

$$\begin{aligned}
&[\![P, Q]\!] \in \Gamma(\wedge^{p+q-1} A),\\
&[\![X, f]\!] = \rho(X)(f),\\
&[\![P, Q]\!] = (-1)^{pq}[\![Q, P]\!], \qquad (1.20)\\
&[\![P, Q \wedge R]\!] = [\![P, Q]\!] \wedge R + (-1)^{q(p+1)} Q \wedge [\![P, R]\!],\\
&(-1)^{pr}[\![[\![P, Q]\!], R]\!] + (-1)^{qr}[\![[\![R, P]\!], Q]\!] + (-1)^{pq}[\![[\![Q, R]\!], P]\!] = 0,
\end{aligned}$$

for $X \in \Gamma(A)$, $f \in C^\infty(M,\mathbb{R})$, $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^r A)$ (see [110]).

Remark 1.7 The definition of Schouten bracket considered here is the one given in [110] (see also [3, 73]). Some authors (see, for instance, [61]) define the Schouten bracket in another way. In fact, the relation between the Schouten bracket $[\![\ ,\ ]\!]'$ in the sense of [61] and the Schouten bracket $[\![\ ,\ ]\!]$ in the sense of [110] is the following one:

$$[\![P,Q]\!]' = (-1)^{p+1}[\![P,Q]\!],$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$.

On the other hand, imitating the de Rham differential on the space $\Omega^*(M)$, we define the *differential of the Lie algebroid* A, $d\colon \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$, as follows:

$$\begin{aligned} d\mu(X_0,\dots,X_k) &= \sum_{i=0}^{k}(-1)^i\rho(X_i)(\mu(X_0,\dots,\hat{X}_i,\dots,X_k)) \\ &+ \sum_{i<j}(-1)^{i+j}\mu([\![X_i,X_j]\!],X_0,..,\hat{X}_i,..,\hat{X}_j,..,X_k), \end{aligned} \tag{1.21}$$

for $\mu \in \Gamma(\wedge^k A^*)$ and $X_0,\dots,X_k \in \Gamma(A)$.

Moreover, since $d^2 = 0$, we have the corresponding cohomology spaces. This cohomology is the *Lie algebroid cohomology with trivial coefficients* (see [82]).

Using the above definitions, it follows that a 1-cochain $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle if and only if

$$\phi_0[\![X,Y]\!] = \rho(X)(\phi_0(Y)) - \rho(Y)(\phi_0(X)), \tag{1.22}$$

for all $X, Y \in \Gamma(A)$.
In addition, if $X \in \Gamma(A)$, we can define the Lie derivative of a multi-section of the dual bundle A^* as the commutator of the differential and the contraction by X, that is, $\mathcal{L}_X = d\circ i_X + i_X \circ d$.

1.2.2 Examples of Lie algebroids

Next, we will consider some examples of Lie algebroids.

1.- *Real Lie algebras of finite dimension*
Let $\mathfrak{g}$ be a real Lie algebra of finite dimension. Then, it is clear that $\mathfrak{g}$ is a Lie algebroid over a single point. The differential (respectively, the Schouten bracket) on $\mathfrak{g}$ is just the algebraic differential (respectively, the algebraic Schouten bracket) on $\mathfrak{g}$.

2.- *The tangent bundle*
Let TM be the tangent bundle of a manifold M. Then, the triple $(TM, [\, ,\,],$ $Id)$ is a Lie algebroid over M, where $Id : TM \to TM$ is the identity map. In this case, the differential (respectively, the Schouten bracket) of TM is just the de Rham differential d_0 on $\Omega^*(M) = \oplus_k \Omega^k(M)$ (respectively, the usual Schouten-Nijenhuis bracket on $\mathcal{V}^*(M) = \oplus_k \mathcal{V}^k(M)$).

3.- *The Lie algebroid* $(TM \times \mathbb{R}, [\ ,\ ], \pi)$
If M is a differentiable manifold, we will exhibit a natural Lie algebroid structure on the vector bundle $TM \times \mathbb{R} \to M$. First, we will show some identifications which will be useful in the sequel.

Let $A \to M$ be a vector bundle over M. Then, it is clear that $A \times \mathbb{R}$ is the total space of a vector bundle over M. Moreover, the dual bundle to $A \times \mathbb{R}$ is $A^* \times \mathbb{R}$ and the spaces $\Gamma(\wedge^r(A \times \mathbb{R}))$ and $\Gamma(\wedge^k(A^* \times \mathbb{R}))$ can be identified with $\Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1} A)$ and $\Gamma(\wedge^k A^*) \oplus \Gamma(\wedge^{k-1} A^*)$ in such a way that

$$
\begin{aligned}
&(P, Q)((\mu_1, f_1), \dots, (\mu_r, f_r)) \\
&\qquad = P(\mu_1, \dots, \mu_r) + \sum_{i=1}^{r} (-1)^{i+1} f_i\, Q(\mu_1, \dots, \hat{\mu}_i, \dots, \mu_r), \\
&(\mu, \nu)((X_1, g_1), \dots, (X_k, g_k)) \\
&\qquad = \mu(X_1, \dots, X_k) + \sum_{i=1}^{k} (-1)^{i+1} g_i\, \nu(X_1, \dots, \hat{X}_i, \dots, X_k),
\end{aligned}
\tag{1.23}
$$

for $(P, Q) \in \Gamma(\wedge^r A) \oplus \Gamma(\wedge^{r-1} A)$, $(\mu, \nu) \in \Gamma(\wedge^k A^*) \oplus \Gamma(\wedge^{k-1} A^*)$, $(\mu_i, f_i) \in \Gamma(A^*) \oplus C^\infty(M, \mathbb{R})$ and $(X_j, g_j) \in \Gamma(A) \oplus C^\infty(M, \mathbb{R})$, with $i \in \{1, \dots, r\}$ and

$j \in \{1, \dots, k\}$.
Under these identifications, the contractions and the exterior products are given by

$$\begin{array}{llll} i_{(\mu,\nu)}(P,Q) & = & (i_\mu P + i_\nu Q, (-1)^k i_\mu Q), & \text{if } k \leq r, \\ i_{(\mu,\nu)}(P,Q) & = & 0, & \text{if } k > r, \\ i_{(P,Q)}(\mu,\nu) & = & (i_P\mu + i_Q\nu, (-1)^r i_P\nu), & \text{if } r \leq k, \\ i_{(P,Q)}(\mu,\nu) & = & 0, & \text{if } r > k, \\ (P,Q)\wedge(P',Q') & = & (P\wedge P', Q\wedge P' + (-1)^r P\wedge Q'), & \\ (\mu,\nu)\wedge(\mu',\nu') & = & (\mu\wedge\mu', \nu\wedge\mu' + (-1)^k\mu\wedge\nu'), & \end{array} \tag{1.24}$$

for $(P', Q') \in \Gamma(\wedge^{r'} A) \oplus \Gamma(\wedge^{r'-1} A)$ and $(\mu', \nu') \in \Gamma(\wedge^{k'} A^*) \oplus \Gamma(\wedge^{k'-1} A^*)$.

Now, suppose that A is the tangent bundle TM. Then, the triple $(A \times \mathbb{R} = TM \times \mathbb{R}, [\ ,\ ], \pi)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\ ,\ ]$ is the bracket given by (see [82, 94])

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)), \tag{1.25}$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$. In this case, the dual bundle to $TM \times \mathbb{R}$ is $T^*M \times \mathbb{R}$ and the spaces $\Gamma(\wedge^r(TM \times \mathbb{R}))$ and $\Gamma(\wedge^k(T^*M \times \mathbb{R}))$ can be identified with $\mathcal{V}^r(M) \oplus \mathcal{V}^{r-1}(M)$ and $\Omega^k(M) \oplus \Omega^{k-1}(M)$. Under these identifications, the differential $\tilde{d}_0$ of the Lie algebroid is

$$\tilde{d}_0(\mu, \nu) = (d_0\mu, -d_0\nu) \tag{1.26}$$

and the Schouten bracket $[\, ,\,]$ is given by

$$[(P, Q), (P', Q')] = ([P, P'], (-1)^{r+1}[P, Q'] - [Q, P']), \tag{1.27}$$

for $(\mu, \nu) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$, $(P, Q) \in \mathcal{V}^r(M) \oplus \mathcal{V}^{r-1}(M)$ and $(P', Q') \in \mathcal{V}^{r'}(M) \oplus \mathcal{V}^{r'-1}(M)$. Thus, $(\mu, f) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle if and only if μ is a closed 1-form and f is a constant function. In particular, the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle.

4.- *The Lie algebroid* $(T^*M \times \mathbb{R}, [\![\, , \,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ *associated with a Jacobi manifold* (M, Λ, E)

If $A \to M$ is a vector bundle over M and $P \in \Gamma(\wedge^2 A)$ is a 2-section of A, we will denote by $\#_P : A^* \to A$ the bundle map given by

$$\nu_x(\#_P(\mu_x)) = P(x)(\mu_x, \nu_x), \tag{1.28}$$

for $\mu_x, \nu_x \in A^*_x$. We will also denote by $\#_P : \Gamma(A^*) \to \Gamma(A)$ the corresponding homomorphism of $C^\infty(M, \mathbb{R})$-modules.

Then, a Jacobi manifold (M, Λ, E) has an associated Lie algebroid $(T^*M \times \mathbb{R}, [\![\, , \,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$, where $([\![\, , \,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ are defined by

$$\begin{aligned}
&[\![(\mu, f), (\nu, g)]\!]_{(\Lambda,E)} \\
&\quad = ((\mathcal{L}_0)_{\#_\Lambda(\mu)}\nu - (\mathcal{L}_0)_{\#_\Lambda(\nu)}\mu - d_0(\Lambda(\mu,\nu)) + f(\mathcal{L}_0)_E\nu - g(\mathcal{L}_0)_E\mu \\
&\qquad - i_E(\mu\wedge\nu), \Lambda(\nu,\mu) + \#_\Lambda(\mu)(g) - \#_\Lambda(\nu)(f) + fE(g) - gE(f)),
\end{aligned} \tag{1.29}$$

$$\widetilde{\#}_{(\Lambda,E)}(\mu, f) \quad = \quad \#_\Lambda(\mu) + fE,$$

for $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ (see [56]). For this Lie algebroid, the differential d_* is given by (see [66, 67])

$$d_*(P, Q) = (-[\Lambda, P] + kE\wedge P + \Lambda\wedge Q, [\Lambda, Q] - (k-1)E\wedge Q + [E, P]), \tag{1.30}$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$. Thus, $(X, f) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle if and only if

$$(\mathcal{L}_0)_X\Lambda = E \wedge X + f\Lambda, \qquad \#_\Lambda(d_0 f) = -[E, X].$$

Therefore, the pair $(E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle.

In the particular case when (M, Λ) is a Poisson manifold we recover, by projection on the first factor, the Lie algebroid $(T^*M, [\![\, , \,]\!]_\Lambda, \#_\Lambda)$, where $[\![\, , \,]\!]_\Lambda$ is the bracket of 1-forms defined by (see [3, 14, 30, 110]):

$$[\![\mu, \nu]\!]_\Lambda = (\mathcal{L}_0)_{\#_\Lambda(\mu)}\nu - (\mathcal{L}_0)_{\#_\Lambda(\nu)}\mu - d_0(\Lambda(\mu, \nu)), \tag{1.31}$$

for $\mu, \nu \in \Omega^1(M)$. For this Lie algebroid, the differential is the operator given by

$$d_* P = -[\Lambda, P], \tag{1.32}$$

for $P \in \mathcal{V}^k(M)$. This operator was introduced by Lichnerowicz in [73] to define the *Lichnerowicz-Poisson cohomology.*

Remark 1.8 Let M be a smooth manifold. If $\mu \in \Omega^1(M)$ and $f \in C^\infty(M, \mathbb{R})$, we will denote by $\overline{(\mu, f)}$ the 1-form on $M \times \mathbb{R}$ given by

$$\overline{(\mu, f)} = e^t(\mu + f\, d_0 t).$$

Now, suppose that (Λ, E) is a Jacobi structure on M and that Π is the Poissonization on $M \times \mathbb{R}$ of the Jacobi structure (Λ, E). Then, a direct computation, using (1.29) and (1.31), proves that

$$[\![\overline{(\mu, f)}, \overline{(\nu, g)}]\!]_\Pi = \overline{[\![(\mu, f), (\nu, g)]\!]_{(\Lambda, E)}}, \tag{1.33}$$

for $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

5.- *The Lie algebroid associated with a Nijenhuis operator*
Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid over M and $\mathcal{N} : A \to A$ be a vector bundle morphism over the identity $Id : M \to M$. Let us also denote by $\mathcal{N} : \Gamma(A) \to \Gamma(A)$ the corresponding homomorphism of $C^\infty(M, \mathbb{R})$-modules. We say that $\mathcal{N}$ is a *Nijenhuis operator* on $(A, [\![\,,\,]\!], \rho)$ if it has vanishing Nijenhuis torsion $\mathcal{T}(\mathcal{N})$, where $\mathcal{T}(\mathcal{N})$ is defined by

$$\mathcal{T}(\mathcal{N})(X, Y) = [\![\mathcal{N}X, \mathcal{N}Y]\!] - \mathcal{N}[\![\mathcal{N}X, Y]\!] - \mathcal{N}[\![X, \mathcal{N}Y]\!] + \mathcal{N}^2[\![X, Y]\!],$$

for $X, Y \in \Gamma(A)$. Note that for the usual Lie algebroid structure on the tangent bundle of an arbitrary manifold M, we recover the usual notion of a Nijenhuis operator on M.

If $\mathcal{N}$ is a Nijenhuis operator on $(A, [\![\,,\,]\!], \rho)$ then there exists a deformed Lie algebroid structure $([\![\,,\,]\!]_\mathcal{N}, \rho_\mathcal{N})$ on $A \to M$, where $[\![\,,\,]\!]_\mathcal{N}$ and $\rho_\mathcal{N}$ are given by (see [36, 64])

$$\begin{aligned} [\![X, Y]\!]_\mathcal{N} &= [\![\mathcal{N}X, Y]\!] + [\![X, \mathcal{N}Y]\!] - \mathcal{N}[\![X, Y]\!], \\ \rho_\mathcal{N} &= \rho \circ \mathcal{N}. \end{aligned} \tag{1.34}$$

The differential $d_{\mathcal{N}}$ of the Lie algebroid $(A, [\![\, ,\,]\!]_{\mathcal{N}}, \rho_{\mathcal{N}})$ is

$$d_{\mathcal{N}} = i_{\mathcal{N}} \circ d - d \circ i_{\mathcal{N}}, \tag{1.35}$$

$i_{\mathcal{N}} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$ being the contraction by $\mathcal{N}$ defined by

$$(i_{\mathcal{N}}(\gamma))(X_1, \dots, X_k) = \sum_{i=1}^{k} \gamma(X_1, \dots, \mathcal{N}X_i, \dots, X_k), \tag{1.36}$$

for $\gamma \in \Gamma(\wedge^k A^*)$ and $X_1, \dots, X_k \in \Gamma(A)$.

On the other hand, denote by $\mathcal{N}^* : \Gamma(A^*) \to \Gamma(A^*)$ the adjoint operator of $\mathcal{N} : \Gamma(A) \to \Gamma(A)$ and by $i_{\mathcal{N}^*} : \Gamma(\wedge^r A) \to \Gamma(\wedge^r A)$ the natural extension of $\mathcal{N}$ to the space $\Gamma(\wedge^r A)$ defined by

$$(i_{\mathcal{N}^*}R)(\mu_1, \dots, \mu_r) = \sum_{i=1}^{r} R(\mu_1, \dots, \mathcal{N}^*\mu_i, \dots, \mu_r),$$

for $R \in \Gamma(\wedge^r A)$ and $\mu_1, \dots, \mu_r \in \Gamma(A^*)$. Then, the Schouten bracket $[\![\, ,\,]\!]_{\mathcal{N}}$ of the Lie algebroid $(A, [\![\, ,\,]\!]_{\mathcal{N}}, \rho_{\mathcal{N}})$ is given by

$$[\![P, Q]\!]_{\mathcal{N}} = [\![i_{\mathcal{N}^*}P, Q]\!] + [\![P, i_{\mathcal{N}^*}Q]\!] - i_{\mathcal{N}^*}[\![P, Q]\!],$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^q A)$.

6.- *Action of a Lie algebroid on a smooth map*
Let $(A, [\![\, ,\,]\!], \rho)$ be a Lie algebroid over a manifold M and $\pi : P \to M$ be a smooth map. An action of A on $\pi : P \to M$ is a $\mathbb{R}$-linear map

$$* : \Gamma(A) \to \mathfrak{X}(P), \quad X \in \Gamma(A) \mapsto X^* \in \mathfrak{X}(P),$$

such that:

$$(fX)^* = (f \circ \pi)X^*, \quad [\![X, Y]\!]^* = [X^*, Y^*], \quad \pi_*^p(X^*(p)) = \rho(X(\pi(p))),$$

for $f \in C^\infty(M, \mathbb{R})$, $X, Y \in \Gamma(A)$ and $p \in M$. If $* : \Gamma(A) \to \mathfrak{X}(P)$ is an action of A on $\pi : P \to M$ and $\tau : A \to M$ is the bundle projection then the pullback vector bundle of A over π,

$$\pi^* A = \{(a, p) \in A \times P \,/\, \tau(a) = \pi(p)\},$$

is a Lie algebroid over P with the Lie algebroid structure $(\llbracket\, ,\, \rrbracket_\pi, \rho_\pi)$ which is characterized by

$$\llbracket X, Y \rrbracket_\pi = \llbracket X, Y \rrbracket \circ \pi, \qquad \rho_\pi(X)(p) = X^*(p),$$

for $X, Y \in \Gamma(A)$ and $p \in P$. The triple $(\pi^* A, \llbracket\, ,\, \rrbracket_\pi, \rho_\pi)$ is called the *action Lie algebroid of* A *on* π and it is denoted by $A \ltimes \pi$ or $A \ltimes P$ (see [42]).

7.- *The Lie algebroid of an exact Poisson structure*
An *exact Poisson manifold* (M, Π, Z) is a Poisson manifold (M, Π) with a vector field Z such that $[Z, \Pi] = -\Pi$ (see [24]). We may consider the Lie algebroid structure $(\llbracket\, ,\, \rrbracket_\Pi, \#_\Pi)$ on the vector bundle $T^*M \to M$ induced by the Poisson structure Π (see (1.31)) and the Lie algebroid structure $(\llbracket\, ,\, \rrbracket_Z, \rho_Z)$ on the vector bundle $M \times \mathbb{R} \to M$ induced by the vector field Z, that is, $\llbracket\, ,\, \rrbracket_Z$ and ρ_Z are defined by

$$\llbracket f, g \rrbracket_Z = g\, Z(f) - f\, Z(g), \quad \rho_Z(f) = -f\, Z,$$

for $f, g \in C^\infty(M, \mathbb{R}) \cong \Gamma(M \times \mathbb{R})$.
Moreover, using the homogeneous character of Π with respect to Z, one can introduce a Lie algebroid structure $(\llbracket\, ,\, \rrbracket_{(\Pi,Z)}, \bar{\#}_{(\Pi,Z)})$ on the vector bundle $T^*M \times \mathbb{R} \to M$, where $\llbracket\, ,\, \rrbracket_{(\Pi,Z)}$ and $\bar{\#}_{(\Pi,Z)}$ are given by

$$\begin{aligned}
&\llbracket (\mu, f), (\nu, g) \rrbracket_{(\Pi,Z)} \\
&\quad = ((\mathcal{L}_0)_{\#_\Pi(\mu)}\nu - (\mathcal{L}_0)_{\#_\Pi(\nu)}\mu - d_0(\Pi(\mu,\nu)) - f((\mathcal{L}_0)_Z\nu - \nu) \\
&\qquad + g((\mathcal{L}_0)_Z\mu - \mu), \#_\Pi(\mu)(g) - \#_\Pi(\nu)(f) + g\, Z(f) - f\, Z(g)), \\
&\bar{\#}_{(\Pi,Z)}(\mu, f) \quad = \quad \#_\Pi(\mu) - f\, Z,
\end{aligned} \tag{1.37}$$

for $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ (see [113]).

In addition, using (1.37), we deduce that $(T^*M, \llbracket\, ,\, \rrbracket_\Pi, \#_\Pi)$ and $(M \times \mathbb{R}, \llbracket\, ,\, \rrbracket_Z, \rho_Z)$ are Lie subalgebroids of $(T^*M \times \mathbb{R}, \llbracket\, ,\, \rrbracket_{(\Pi,Z)}, \bar{\#}_{(\Pi,Z)})$. In fact, $(T^*M, \llbracket\, ,\, \rrbracket_\Pi, \#_\Pi)$ and $(M \times \mathbb{R}, \llbracket\, ,\, \rrbracket_Z, \rho_Z)$ form a *matched pair of Lie algebroids* in the sense of Mokri [93].

For the Lie algebroid $(T^*M\times\mathbb{R}, [\![\, ,\,]\!]_{(\Pi,Z)}, \bar{\#}_{(\Pi,Z)})$, the differential d_* is given by

$$d_*(P,Q) = (-[\Pi,P],[\Pi,Q]-[Z,P]-kP), \tag{1.38}$$

for $(P,Q)\in\mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$. Thus, $(X,f)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\cong\Gamma(TM\times\mathbb{R})$ is a 1-cocycle if and only if

$$(\mathcal{L}_0)_X\Pi = 0, \qquad \#_\Pi(d_0 f) = [Z,X]+X.$$

In particular, the pair $(0,1)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$ is a 1-cocycle.

8.- *The Lie algebroid associated with a linear Poisson structure*
Let $\tau : A\to M$ be a vector bundle on a manifold M. Then, it is clear that there exists a bijection between the space $\Gamma(A^*)$ of sections of the dual bundle $\tau^*: A^*\to M$ and the set $L(A)$ of real functions on A which are linear on each fiber

$$\Gamma(A^*)\to L(A), \quad \mu\mapsto\tilde{\mu}.$$

For any section $\mu\in\Gamma(A^*)$ the corresponding linear function $\tilde{\mu}$ on the vector bundle A is given by $\tilde{\mu}(X_p)=\mu(p)(X_p)$, for $X_p\in A_p$. Note that if $f : A\to\mathbb{R}$ is a smooth real function then

$$\begin{aligned} &f \text{ is basic} \Leftrightarrow \Delta_A(f)=0,\\ &f \text{ is linear} \Leftrightarrow \Delta_A(f)=f, \end{aligned} \tag{1.39}$$

where Δ_A denotes the *Liouville* (Euler) vector field on A.

On the other hand, a 2-vector Π on A is *linear* if and only if the induced bracket $\{\, ,\, \}_\Pi$ is closed on linear functions, that is, if $\mu,\nu\in\Gamma(A^*)$ then

$$\{\tilde{\mu},\tilde{\nu}\}_\Pi = i_{(d_0\tilde{\mu}\wedge d_0\tilde{\nu})}\Pi$$

is again a linear function. If Π is a linear 2-vector field on A and $f_M, g_M : M\to\mathbb{R}$ are real smooth functions then

$$\{\tilde{\mu}, f_M\circ\tau\}_\Pi \text{ is a basic function and } \{f_M\circ\tau, g_M\circ\tau\}_\Pi = 0. \tag{1.40}$$

Using the above facts, it is easy to prove that

$$\begin{aligned} \Pi \text{ is linear} \quad &\Leftrightarrow \quad \Pi \text{ is homogeneous with respect to } \Delta_A \\ &\Leftrightarrow \quad (\mathcal{L}_0)_{\Delta_A}\Pi = -\Pi. \end{aligned} \tag{1.41}$$

Now, suppose that Π is a linear Poisson structure on A^* with Poisson bracket $\{\,,\,\}_\Pi$. From (1.40), one may define a Lie algebroid structure $([\![\,,\,]\!]^\Pi, \rho^\Pi)$ on $\tau : A \to M$ which is characterized by

$$\begin{aligned} \widetilde{[\![X, Y]\!]}^{\,\Pi} &= \{\tilde{X}, \tilde{Y}\}_\Pi, \\ \rho^\Pi(X)(f_M) \circ \tau^* &= \{\tilde{X}, f_M \circ \tau^*\}_\Pi, \end{aligned} \tag{1.42}$$

for $X, Y \in \Gamma(A)$ and $f_M \in C^\infty(M, \mathbb{R})$, $\tau^* : A^* \to M$ being the canonical projection (see [14, 15]). Conversely, if A is a vector bundle over M which admits a Lie algebroid structure $([\![\,,\,]\!], \rho)$ then one may define a linear Poisson structure Π_{A^*} on the dual bundle A^* in such a way that (1.42) holds.

The local expression of Π_{A^*} is given as follows. Let U be an open coordinate neighbourhood of M with coordinates $(x^1, \ldots, x^m)$ and $\{e_i\}_{i=1,\ldots,n}$ a local basis of sections of $\tau : A \to M$ in U. Then, $\tau^{-1}(U)$ is an open coordinate neighbourhood of A with coordinates (x^i, v_j) such that $v_j = \tilde{e}_j$, for all j. In these coordinates the structure functions and the components of the anchor map are

$$[\![e_i, e_j]\!] = \sum_{k=1}^{n} c_{ij}^k e_k, \qquad \rho(e_i) = \sum_{l=1}^{m} \rho_i^l \frac{\partial}{\partial x^l}, \quad i, j \in \{1, \ldots, n\}, \tag{1.43}$$

with $c_{ij}^k, \rho_i^l \in C^\infty(U, \mathbb{R})$, and the Poisson structure Π_{A^*} is given by

$$\Pi_{A^*} = \sum_{i<j} \sum_{k} c_{ij}^k v_k \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial v_j} + \sum_{i,l} \rho_i^l \frac{\partial}{\partial v_i} \wedge \frac{\partial}{\partial x^l}. \tag{1.44}$$

Note that the Liouville vector field is given by

$$\Delta_A = \sum_{i=1}^{n} v_i \frac{\partial}{\partial v_i}. \tag{1.45}$$

Remark 1.9 Let $(A, [\![\, ,\,]\!], \rho)$ be a Lie algebroid and ϕ_0 be a section of the dual bundle A^* to A. Then, one may consider the vertical lift $\phi_0^{\mathbf{v}} \in \mathfrak{X}(A^*)$ of ϕ_0 and, using (1.42) and the fact that

$$\phi_0^{\mathbf{v}}(\tilde{X}) = \phi_0(X) \circ \tau^*, \text{ for } X \in \Gamma(A),$$

we deduce that

$$((\mathcal{L}_0)_{\phi_0^{\mathbf{v}}} \Pi_{A^*})(d_0\tilde{X}, d_0\tilde{Y}) = -d\phi_0(X, Y) \circ \tau^*,$$

for $X, Y \in \Gamma(A)$, d being the differential of the Lie algebroid $(A, [\![\, ,\,]\!], \rho)$. Thus,

$$(\mathcal{L}_0)_{\phi_0^{\mathbf{v}}} \Pi_{A^*} = 0 \Leftrightarrow \phi_0 \text{ is a 1-cocycle.} \tag{1.46}$$

Examples 1.10 1.- Let $(\mathfrak{g}, [\, ,\,]_{\mathfrak{g}})$ be a real Lie algebra of dimension n. Then, $\mathfrak{g}$ is a Lie algebroid over a point. Moreover, if $\Pi_{\mathfrak{g}^*}$ is the linear Poisson structure on $\mathfrak{g}^*$, using (1.5) and (1.44), we have that $\Pi_{\mathfrak{g}^*}$ is the well known Lie-Poisson structure.

2.- Let $(TM, [\, ,\,], Id)$ be the trivial Lie algebroid. From (1.43) and (1.44), it follows that the linear Poisson structure Π_{T^*M} on T^*M is just the *canonical symplectic structure*, that is,

$$\Pi_{T^*M} = \sum_{i=1}^{m} \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

$(q^1, \dots, q^m, p_1, \dots, p_m)$ being fibred coordinates on T^*M.

3.- Let (M, Π) be a Poisson manifold and $(T^*M, [\![\, ,\,]\!]_\Pi, \#_\Pi)$ be the associated cotangent Lie algebroid. From (1.31), (1.43) and (1.44), we obtain that the induced Poisson structure on TM is the *complete lift* Π^c to TM of Π (see [15]).

4.- The triple $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\ ,\ ]$ is the bracket given by (1.25). In this case, the Poisson structure $\Pi_{T^*M\times\mathbb{R}}$ on $T^*M \times \mathbb{R}$ is just the *canonical cosymplectic structure* of $T^*M \times \mathbb{R}$ (see [1, 7]), that is, $\Pi_{T^*M\times\mathbb{R}} = \Pi_{T^*M}$.

5.- Let (M,Λ,E) be a Jacobi manifold and $(T^*M\times\mathbb{R},[\![\,,\,]\!]_{(\Lambda,E)},\widetilde{\#}_{(\Lambda,E)})$ the associated Lie algebroid. A direct computation, using (1.29), (1.43) and (1.44), shows that the Poisson structure $\Pi_{TM\times\mathbb{R}}$ is

$$\Pi_{TM\times\mathbb{R}} = \Lambda^c - t\Lambda^{\mathbf{v}} - E^{\mathbf{v}}\wedge\Delta_{TM} + \frac{\partial}{\partial t}\wedge E^c,$$

where Λ^c and E^c (respectively, $\Lambda^{\mathbf{v}}$ and $E^{\mathbf{v}}$) are the complete (respectively, vertical) lift to TM of Λ and E and t is the usual coordinate on $\mathbb{R}$.

6.- Let $(A,[\![\,,\,]\!],\rho)$ be a Lie algebroid over a manifold M and $\mathcal{N}$ be a Nijenhuis operator on A. Denote by $([\![\,,\,]\!]_{\mathcal{N}},\rho_{\mathcal{N}})$ the Lie algebroid structure on A given by (1.34), by Π_{A^*} the linear Poisson structure on A^* associated with the Lie algebroid $(A,[\![\,,\,]\!],\rho)$ and by $\mathcal{J}_{A^*}(\mathcal{N})$ the vector field on A^* defined by

$$\mathcal{J}_{A^*}(\mathcal{N})(\mu_p) = (\mathcal{N}^*(\mu_p))^{\mathbf{v}},$$

for $\mu_p\in A^*_p$, where $\mathcal{N}^*: A^*\to A^*$ is the adjoint operator of $\mathcal{N}$ and $\mathbf{v}: A^*_p\to T_{\mu_p}(A^*_p)$ is the vertical lift. Then,

$$\Pi^{\mathcal{N}}_{A^*} = (\mathcal{L}_0)_{\mathcal{J}_{A^*}(\mathcal{N})}\Pi_{A^*},$$

$\Pi^{\mathcal{N}}_{A^*}$ being the linear Poisson structure on A^* associated with the Lie algebroid structure $([\![\,,\,]\!]_{\mathcal{N}},\rho_{\mathcal{N}})$ (see [36]).

7.- Let (M,Π,Z) be an exact Poisson manifold and $(T^*M\times\mathbb{R},[\![\,,\,]\!]_{(\Pi,Z)},\bar{\#}_{(\Pi,Z)})$ be the associated Lie algebroid. From (1.37), (1.43) and (1.44), we obtain that the linear Poisson structure $\Pi_{TM\times\mathbb{R}}$ on $TM\times\mathbb{R}$ is

$$\Pi_{TM\times\mathbb{R}} = \Pi^c + (Z^c - \Delta_{TM})\wedge\frac{\partial}{\partial t}.$$

9.- *The tangent Lie algebroid*

Assume that $\tau: A\to M$ is a Lie algebroid over a manifold M and that $p: A^*\times_M A\to\mathbb{R}$ is the natural pairing. Then, TA and TA^* are vector bundles over TM and p induces a non-degenerate pairing $TA^*\times_{TM}TA\to\mathbb{R}$. Thus, we get an isomorphism between the vector bundles $TA\to TM$ and $(TA^*)^*\to TM$. Therefore, the dual bundle to $TA\to TM$ may be identified with the vector bundle $TA^* \to TM$. On the other hand, denote by Π_{A^*} the linear Poisson structure on A^* induced by the Lie algebroid A. Then, it is easy to prove that the complete lift $\Pi^c_{A^*}$ of Π_{A^*} to TA^* is a linear Poisson structure on the vector bundle $TA^* \to TM$. Consequently, the vector bundle $TA \to TM$ is a Lie algebroid which is called the *tangent Lie algebroid* to A (for more details, see [16, 37, 83]).

1.2.3 Lie algebroid morphisms

Let $(A, [\![\, , \,]\!], \rho)$ (respectively, $(A', [\![\, , \,]\!]', \rho')$) be a Lie algebroid over a manifold M (respectively, M') and suppose that $\Psi : A \to A'$ is a vector bundle morphism over the map $\Psi_0 : M \to M'$. Then, the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\quad\Psi\quad} & A' \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad\Psi_0\quad} & M' \end{array}$$

Now, if $X \in \Gamma(A)$ then

$$\Psi \circ X = \sum_i f_i (X'_i \circ \Psi_0), \tag{1.47}$$

for suitable $f_i \in C^\infty(M, \mathbb{R})$ and $X'_i \in \Gamma(A')$. We refer to a relation (1.47) as a Ψ-*decomposition of* X.

The pair (Ψ, Ψ_0) is said to be a *Lie algebroid morphism* if

$$\rho' \circ \Psi = T\Psi_0 \circ \rho, \tag{1.48}$$

$$\begin{array}{rcl} \Psi \circ [\![X, Y]\!] & = & \displaystyle\sum_{i,j} f_i \, g_j ([\![X'_i, Y'_j]\!]' \circ \Psi_0) + \sum_j \rho(X)(g_j)(Y'_j \circ \Psi_0) \\ & & \displaystyle - \sum_i \rho(Y)(f_i)(X'_i \circ \Psi_0), \end{array} \tag{1.49}$$

for $X, Y \in \Gamma(A)$, where $T\Psi_0 : TM \to TM'$ is the tangent map of Ψ_0 and

$$\Psi \circ X = \sum_i f_i (X_i' \circ \Psi_0), \qquad \Psi \circ Y = \sum_j g_j (Y_j' \circ \Psi_0),$$

are Ψ-decompositions of X and Y, respectively. The right-hand side of equation (1.49) is independent of the choice of the Ψ-decompositions of X and Y (for more details, see [42]).

If $M = M'$, Ψ_0 is the identity map and $X \in \Gamma(A)$ then $\Psi \circ X$ is a section of A' and (1.49) is equivalent to the condition

$$\Psi \circ [\![X, Y]\!] = [\![\Psi \circ X, \Psi \circ Y]\!]', \tag{1.50}$$

for $X, Y \in \Gamma(A)$.

1.3 Lie groupoids. Examples

The global objects corresponding to Lie algebroids are Lie groupoids. In this last Section of Chapter 1, we recall the definition of a Lie groupoid and some generalities about them are explained. We also discuss some examples which will be interesting in the sequel.

1.3.1 Lie groupoids

A *groupoid* consists of two sets G and M, called respectively the *groupoid* and the *base*, together with two maps α and β from G to M, called respectively the *source* and *target* projections, a map $\epsilon : M \to G$, called the *inclusion*, a *partial multiplication* $m : G^{(2)} = \{(g, h) \in G \times G / \alpha(g) = \beta(h)\} \to G$ and a map $\iota : G \to G$, called the *inversion*, satisfying the following conditions:

i) $\alpha(m(g, h)) = \alpha(h)$ and $\beta(m(g, h)) = \beta(g)$, for all $(g, h) \in G^{(2)}$,

ii) $m(g, m(h, k)) = m(m(g, h), k)$, for all $g, h, k \in G$ such that $\alpha(g) = \beta(h)$ and $\alpha(h) = \beta(k)$,

iii) $\alpha(\epsilon(x)) = x$ and $\beta(\epsilon(x)) = x$, for all $x \in M$, *iv)* $m(g,\epsilon(\alpha(g))) = g$ and $m(\epsilon(\beta(g)),g) = g$, for all $g \in G$,

v) $m(g,\iota(g)) = \epsilon(\beta(g))$ and $m(\iota(g),g) = \epsilon(\alpha(g))$, for all $g \in G$.

A groupoid G over a base M will be denoted by $G \rightrightarrows M$.

If G and M are manifolds, $G \rightrightarrows M$ is a *Lie groupoid* if:

i) α and β are differentiable submersions.

ii) m, ϵ and ι are differentiable maps.

From *i)* and *ii)*, it follows that m is a submersion, ϵ is an immersion and ι is a diffeomorphism. In fact, $\iota^2 = Id$.

From now on, we will usually write gh for $m(g,h)$ and g^{-1} for $\iota(g)$. Moreover, if $x \in M$ then $G_x = \alpha^{-1}(x)$ (resp., $G^x = \beta^{-1}(x)$) will be said the *α-fiber* (resp., the *β-fiber*) of x. Furthermore, since ϵ is an inmersion, we will identify M with $\epsilon(M)$.

Next, we will recall some notions related with Lie groupoids which will be useful in the following (for more details, see [82]).

Definition 1.11 *Let $G \rightrightarrows M$ be a Lie groupoid over a manifold M. For $U \subseteq M$ open, a* local bisection *(or* local admissible section*) of G on U is a smooth map $\mathcal{K} : U \to G$ which is right-inverse to β and for which $\alpha \circ \mathcal{K} : U \to \alpha(\mathcal{K}(U))$ is a diffeomorphism from U to the open set $\alpha(\mathcal{K}(U))$ in M. If $U = M$, $\mathcal{K}$ is a global bisection or simply a bisection.*

The existence of local bisections through any point $g \in G$ is always guaranteed.

If $\mathcal{K} : U \to G$ is a local bisection with $V = (\alpha \circ \mathcal{K})(U)$, the local *left-translation* and *right-translation induced by* $\mathcal{K}$ are the maps $L_{\mathcal{K}} : \beta^{-1}(V) \to \beta^{-1}(U)$ and $R_{\mathcal{K}} : \alpha^{-1}(U) \to \alpha^{-1}(V)$ defined by

$$L_{\mathcal{K}}(g) = \mathcal{K}((\alpha \circ \mathcal{K})^{-1}(\beta(g)))\, g, \quad R_{\mathcal{K}}(h) = h\, \mathcal{K}(\alpha(h)), \tag{1.51}$$

for $g \in \beta^{-1}(V)$ and $h \in \alpha^{-1}(U)$.

Remark 1.12 If $y_0 \in U$ and $\mathcal{K}(y_0) = g_0$, $\alpha(g_0) = x_0$ then the restriction of $L_{\mathcal{K}}$ to G^{x_0} is the *left-translation by* g_0

$$L_{g_0} : G^{x_0} \to G^{y_0}, \quad h \mapsto L_{g_0}(h) = g_0 h.$$

In a similar way, the restriction of $R_{\mathcal{K}}$ to G_{y_0} is the *right-translation by* g_0

$$R_{g_0} : G_{y_0} \to G_{x_0}, \quad g \mapsto R_{g_0}(g) = g g_0.$$

A multivector field P on G is said to be *left-invariant* (respectively, *right-invariant*) if it is tangent to the fibers of β (respectively, α) and $P(gh) = (L_{\mathcal{K}})^h_*(P(h))$ (respectively, $P(gh) = (R_{\mathcal{K}})^g_*(P(g)))$ for $(g,h) \in G^{(2)}$ and $\mathcal{K} : U \to G$ any local bisection through h (respectively, g). If P and Q are two left-invariant (respectively, right-invariant) multivector fields on G then $[P,Q]$ is again left-invariant (respectively, right-invariant).

Now, we will recall the definition of the Lie algebroid associated with a Lie groupoid.

Suppose that $G \rightrightarrows M$ is a Lie groupoid. Then, we may consider the vector bundle $AG \to M$, whose fiber at a point $x \in M$ is $A_x G = T_{\epsilon(x)} G^x$. It is easy to prove that there exists a bijection between the space $\Gamma(AG)$ and the set of left-invariant (respectively, right-invariant) vector fields on G. If X is a section of AG, the corresponding left-invariant (respectively, right-invariant) vector field on G will be denoted by $\overleftarrow{X}$ (respectively, $\overrightarrow{X}$). Using the above facts, we may introduce a Lie algebroid structure $(\llbracket\,,\,\rrbracket, \rho)$ on AG, which is defined by

$$\overleftarrow{\llbracket X, Y \rrbracket} = [\overleftarrow{X}, \overleftarrow{Y}], \quad \rho(X)(x) = \alpha_*^{\epsilon(x)}(X(x)), \tag{1.52}$$

for $X, Y \in \Gamma(AG)$ and $x \in M$.

Remark 1.13 There exists a bijection between the space $\Gamma(\wedge^k(AG))$ and the set of left-invariant (respectively, right-invariant) k-vector fields. If P is a section of $\wedge^k(AG)$, we will denote by $\overleftarrow{P}$ (respectively, $\overrightarrow{P}$) the corresponding left-invariant (respectively, right-invariant) k-vector field on G. Moreover, if $P, Q \in \Gamma(\wedge^*(AG))$, we have that

$$\overleftarrow{\llbracket P, Q \rrbracket} = [\overleftarrow{P}, \overleftarrow{Q}], \quad \overrightarrow{\llbracket P, Q \rrbracket} = -[\overrightarrow{P}, \overrightarrow{Q}], \quad [\overrightarrow{P}, \overleftarrow{Q}] = 0. \tag{1.53}$$

Given two Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$, a *morphism of Lie groupoids* is a smooth map $\Phi : G \to G'$ such that if $(g, h) \in G^{(2)}$ then $(\Phi(g), \Phi(h)) \in G'^{(2)}$ and $\Phi(gh) = \Phi(g)\Phi(h)$. A morphism of Lie groupoids $\Phi : G \to G'$ induces a smooth map $\Phi_0 : M \to M'$ in such a way that $\alpha' \circ \Phi = \Phi_0 \circ \alpha$, $\beta' \circ \Phi = \Phi_0 \circ \beta$ and $\Phi \circ \epsilon = \epsilon' \circ \Phi_0$, α, β and ϵ (resp., α', β' and ϵ') being the projections and the inclusion in the Lie groupoid $G \rightrightarrows M$ (resp., $G' \rightrightarrows M'$). If (Φ, Φ_0) is a morphism between the Lie groupoids $G \rightrightarrows M$ and $G' \rightrightarrows M'$ and $AG \to M$ (respectively, $AG' \to M'$) is the Lie algebroid of G (respectively, G') then (Φ, Φ_0) induces, in a natural way, a morphism $(A(\Phi), \Phi_0)$ between the Lie algebroids AG and AG' (see [42, 82]).

1.3.2 Examples of Lie groupoids

1.- *Lie groups*
Any Lie group G is a Lie groupoid over $\{\mathfrak{e}\}$, the identity element of G. The Lie algebroid associated with G is just the Lie algebra $\mathfrak{g}$ of G.

2.- *The banal groupoid*
Let M be a differentiable manifold. The product manifold $M \times M$ is a Lie groupoid over M in the following way: α is the projection onto the second factor and β is the projection onto the first factor; $\epsilon(x) = (x, x)$ for all $x \in M$ and $m((x, y), (y, z)) = (x, z)$. $M \times M \rightrightarrows M$ is called the *banal groupoid*. The Lie algebroid associated with the banal groupoid is the tangent bundle TM of M.

3.- *The direct product of Lie groupoids*
If $G_1 \rightrightarrows M_1$ and $G_2 \rightrightarrows M_2$ are Lie groupoids, then $G_1 \times G_2 \rightrightarrows M_1 \times M_2$ is a Lie groupoid in a natural way.

4.- *Action groupoids*
Let $G \rightrightarrows M$ be a Lie groupoid and $\pi : P \to M$ be a smooth map. If $P * G = \{(p, g) \in P \times G \,/\, \pi(p) = \beta(g)\}$ then a right action of G on π is a smooth map

$$P * G \to P, \quad (p, g) \mapsto p \cdot g,$$

which satisfies the following relations

$$\pi(p\cdot g)=\alpha(g),\ \text{for all } (p,g)\in P*G,$$
$$(p\cdot g)\cdot h=p\cdot(gh),\ \text{for all } (g,h)\in G^{(2)} \text{ and } (p,g)\in P*G,$$
$$p\cdot\epsilon(\pi(p))=p,\ \text{for all } p\in P.$$

Given such an action one constructs the *action groupoid* $P*G\rightrightarrows P$ by defining

$$\alpha'(p,g)=p\cdot g,\quad \beta'(p,g)=p,$$
$$m'((p,g),(q,h))=(p,gh),\ \text{if } q=p\cdot g,$$
$$\epsilon'(p)=(p,\epsilon(\pi(p))),\quad \iota'(p,g)=(p\cdot g,g^{-1}).$$

Now, if $p\in P$, we consider the map $p\,\cdot:G^{\pi(p)}\to P$ given by

$$p\,\cdot(g)=p\cdot g.$$

Then, if AG is the Lie algebroid of G, the $\mathbb{R}$-linear map

$$*:\Gamma(AG)\to\mathfrak{X}(P),\quad X\in\Gamma(AG)\mapsto X^*\in\mathfrak{X}(P),$$

defined by

$$X^*(p)=(p\,\cdot)_*^{\epsilon(\pi(p))}(X(\pi(p))),\tag{1.54}$$

for all $p\in P$, induces an action of AG on $\pi:P\to M$. In addition, the Lie algebroid associated with the Lie groupoid $P*G\rightrightarrows P$ is the action Lie algebroid $AG\ltimes\pi$ (for more details, see [42]).

5.- *The tangent groupoid*

Let $G\rightrightarrows M$ be a Lie groupoid. Then, the tangent bundle TG is a Lie groupoid over TM. The projections α^T, β^T, the partial multiplication $\oplus_{TG}$, the inclusion ϵ^T and the inversion ι^T are defined by

$$\begin{aligned}
&\alpha^T(X_g)=\alpha_*^g(X_g),\ \text{for } X_g\in T_gG,\\
&\beta^T(Y_h)=\beta_*^h(Y_h),\ \text{for } Y_h\in T_hG,\\
&X_g\oplus_{TG}Y_h=m_*^{(g,h)}(X_g,Y_h),\ \text{if } \alpha^T(X_g)=\beta^T(Y_h),\\
&\epsilon^T(X_x)=\epsilon_*^x(X_x),\ \text{for } X_x\in T_xM,\\
&\iota^T(X_g)=\iota_*^g(X_g),\ \text{for } X_g\in T_gG.
\end{aligned}\tag{1.55}$$

In [120] it has been given an explicit expression for the multiplication $\oplus_{TG}$. If $x=\alpha(g)=\beta(h)$ and $\alpha^T(X_g)=\beta^T(X_h)=W_x\in T_xM$, then

$$X_g\oplus_{TG}Y_h=(L_{\mathcal{X}})_*^h(Y_h)+(R_{\mathcal{Y}})_*^g(X_g)-(L_{\mathcal{X}})_*^h((R_{\mathcal{Y}})_*^{\epsilon(x)}(\epsilon_*^x(W_x))), \quad (1.56)$$

where $\mathcal{X},\mathcal{Y}$ are any (local) bisections of G with $\mathcal{X}(x)=g$ and $\mathcal{Y}(x)=h$. If $AG\to M$ is the Lie algebroid of $G\rightrightarrows M$, then the tangent Lie algebroid $TAG\to TM$ is just the Lie algebroid associated with the tangent groupoid $TG\rightrightarrows TM$ (for more details, see [83]).

Remark 1.14 If G is a Lie group then, from (1.56), it follows that

$$X_g\oplus_{TG}Y_h=(L_g)_*^h(Y_h)+(R_h)_*^g(X_g), \quad (1.57)$$

for $X_g\in T_gG$ *and* $Y_h\in T_hG$.

6.- *The cotangent groupoid*
Let $G\rightrightarrows M$ be a Lie groupoid. If A^*G is the dual bundle to AG then the cotangent bundle T^*G is a Lie groupoid over A^*G. The projections $\tilde\alpha$ and $\tilde\beta$, the partial multiplication $\oplus_{T^*G}$, the inclusion $\tilde\epsilon$ and the inversion $\tilde\iota$ are defined as follows,

$$\begin{aligned}
&\tilde\alpha(\mu_g)(X)=\mu_g((L_g)_*^{\epsilon(\alpha(g))}(X)),\ \text{for } \mu_g\in T_g^*G \text{ and } X\in A_{\alpha(g)}G,\\
&\tilde\beta(\nu_h)(Y)=\nu_h((R_h)_*^{\epsilon(\beta(h))}(Y-\epsilon_*^{\beta(h)}(\alpha_*^{\epsilon(\beta(h))}(Y)))),\\
&\qquad\qquad\qquad\qquad \text{for } \nu_h\in T_h^*G \text{ and } Y\in A_{\beta(h)}G,\\
&(\mu_g\oplus_{T^*G}\nu_h)(X_g\oplus_{TG}Y_h)=\mu_g(X_g)+\nu_h(Y_h),\\
&\qquad\qquad\qquad\qquad \text{for } (X_g,Y_h)\in T_{(g,h)}G^{(2)},\\
&\tilde\epsilon(\mu_x)(X_{\epsilon(x)})=\mu_x(X_{\epsilon(x)}-\epsilon_*^x(\beta_*^{\epsilon(x)}(X_{\epsilon(x)}))),\\
&\qquad\qquad\qquad\qquad \text{for } \mu_x\in A_x^*G \text{ and } X_{\epsilon(x)}\in T_{\epsilon(x)}G,\\
&\tilde\iota(\mu_g)(X_{g^{-1}})=-\mu_g(\iota_*^{g^{-1}}(X_{g^{-1}})),\ \text{for } \mu_g\in T_g^*G \text{ and } X_{g^{-1}}\in T_{g^{-1}}G.
\end{aligned} \quad (1.58)$$

Note that $\tilde\epsilon(A^*G)$ is just the conormal bundle of $M\cong\epsilon(M)$ as a submanifold of G.
On the other hand, since A^*G is a Poisson manifold, the cotangent bundle $T^*(A^*G)$ is a Lie algebroid. In fact, the Lie algebroid of the cotangent Lie

groupoid $T^*G \rightrightarrows A^*G$ may be identified with $T^*(A^*G)$ (for more details, see [14, 83]).

Remark 1.15 If G is a Lie group and $\mu_g \in T_g^*G$, $\nu_h \in T_h^*G$ satisfy $\tilde{\alpha}(\mu_g) = \tilde{\beta}(\nu_h)$ then, from (1.57), it follows that

$$\mu_g \oplus_{T^*G} \nu_h = \frac{1}{2}\Big\{((R_{h^{-1}})_*^{gh})^*(\mu_g) + ((L_{g^{-1}})_*^{gh})^*(\nu_h)\Big\}. \tag{1.59}$$

CHAPTER 2

Jacobi algebroids, homogeneous Jacobi structures and its characteristic foliation

In this Chapter, we consider a particular class of Jacobi structures on vector bundles which includes homogeneous (linear) Poisson structures. We obtain a correspondence of this type of structures with Jacobi algebroid structures, a class generalizing Lie algebroid structures. We also discuss some examples and applications. Finally, we prove that the leaves of the characteristic foliation of this type of Jacobi structures on a vector space are the orbits of an action of a Lie group on the vector space and we describe such an action.

2.1 Homogeneous Jacobi structures

In this Section, we will describe a particular class of Jacobi structures on vector bundles and we will give some of its properties.

Definition 2.1 *Let $\tau : A \to M$ be a vector bundle and (Λ, E) be a Jacobi structure on A. (Λ, E) is said to be* homogeneous *if Λ and E are homogeneous with respect to Liouville vector field Δ_A, that is,*

$$(\mathcal{L}_0)_{\Delta_A}\Lambda = -\Lambda, \qquad (\mathcal{L}_0)_{\Delta_A}E = -E.$$

(Λ, E) *is said to be* linear *if the Jacobi bracket of linear functions is again a linear function.*

Now, we prove the following characterizations.

Theorem 2.2 *A Jacobi structure* (Λ, E) *is homogeneous if and only if the first-order differential operator* $D = \Delta_A - Id$ *acts as a derivation of the corresponding Jacobi bracket, that is,*

$$D(\{f,g\}_{(\Lambda,E)}) = \{D(f),g\}_{(\Lambda,E)} + \{f,D(g)\}_{(\Lambda,E)},$$

for $f, g \in C^\infty(M,\mathbb{R})$.

Proof: A direct computation proves that

$$\begin{aligned} &D(\{f,g\}_{(\Lambda,E)}) - \{Df,g\}_{(\Lambda,E)} - \{f,Dg\}_{(\Lambda,E)} \\ &\quad = ((\mathcal{L}_0)_{\Delta_A}\Lambda + \Lambda)(d_0f, d_0g) + f((\mathcal{L}_0)_{\Delta_A}E + E)(g) - g((\mathcal{L}_0)_{\Delta_A}E + E)(f), \end{aligned}$$

which implies the result. QED

Before characterizing homogeneous Jacobi structures in terms of linear brackets we consider the canonical family $\mathcal{V}(A) = \{X^{\mathbf{v}} \,/\, X \in \Gamma(A)\}$ of vertical lifts of sections of A. We note that $X^{\mathbf{v}}$ is always a homogeneous vector field with respect to Δ_A, for all $X \in \Gamma(A)$. In fact, if $(x^1, \dots, x^m)$ are local coordinates on an open subset U of M and $\{e_1, \dots, e_n\}$ is a local basis of sections of A in U such that $X = \sum_{i=1}^{n} X^i e_i$ then $(x^i, v_j = \tilde{e}_j)$ are local coordinates on $\tau^{-1}(U)$ and

$$X^{\mathbf{v}} = \sum_{i=1}^{n} X^i \frac{\partial}{\partial v_i}. \tag{2.1}$$

Theorem 2.3 *Let* (Λ, E) *be a Jacobi structure on a vector bundle* A*. Then, the following statements are equivalent:*

i) (Λ, E) *is homogeneous;*

ii) The Jacobi structure (Λ, E) *is linear and the bracket of a linear function and the constant function 1 is a basic function;*

iii) $E \in \mathcal{V}(A)$ *and there exists a linear Poisson structure* Π_A *on* A *such that*

$$\Lambda = \Pi_A + E \wedge \Delta_A. \tag{2.2}$$

Proof: *i)* $\Rightarrow$ *ii)* If μ, ν are sections of A^* then, from (1.39) and Theorem 2.2, it follows that

$$D(\{\tilde{\mu}, \tilde{\nu}\}_{(\Lambda,E)}) = 0,$$

which implies that $\{\tilde{\mu}, \tilde{\nu}\}_{(\Lambda,E)}$ is linear.
On the other hand, since E is homogeneous, we obtain that

$$\Delta_A(E(\tilde{\mu})) = 0$$

and thus $E(\tilde{\mu}) = \{1, \tilde{\mu}\}_{(\Lambda,E)}$ is a basic function (see (1.39)).

ii) $\Rightarrow$ *iii)* Let f be a basic function. If μ is a section of A^*, then the functions

$$\{1, \tilde{\mu}\}_{(\Lambda,E)}, \qquad \{1, f\tilde{\mu}\}_{(\Lambda,E)} = E(f)\tilde{\mu} + f\{1, \tilde{\mu}\}_{(\Lambda,E)}$$

are basic. Therefore, $E(f) = 0$. Consequently, $E \in \mathcal{V}(A)$ (note that $E(\tilde{\mu})$ is a basic function, for all $\mu \in \Gamma(A^*)$).
Next, we will prove that Λ is linear. For $\mu, \nu \in \Gamma(A^*)$, we have

$$\{\tilde{\mu}, \tilde{\nu}\}_\Lambda = \Lambda(d_0\tilde{\mu}, d_0\tilde{\nu}) = \{\tilde{\mu}, \tilde{\nu}\}_{(\Lambda,E)} - \tilde{\mu}\, E(\tilde{\nu}) + \tilde{\nu}\, E(\tilde{\mu})$$

and, since $E(\tilde{\mu})$ and $E(\tilde{\nu})$ are basic functions, we conclude that $\Lambda(d_0\tilde{\mu}, d_0\tilde{\nu})$ is a linear function. This implies that (see (1.41))

$$(\mathcal{L}_0)_{\Delta_A}\Lambda = -\Lambda$$

and thus, since $E \in \mathcal{V}(A)$, we deduce that

$$[\Lambda - E \wedge \Delta_A, \Lambda - E \wedge \Delta_A] = 0,$$

that is, $\Pi_A = \Lambda - E \wedge \Delta_A$ is a Poisson structure on A. Finally, from (1.41) and using that Λ is linear and the fact that $E \in \mathcal{V}(A)$, we obtain that Π_A is a linear 2-vector on A.

iii) $\Rightarrow$ *i)* If $E \in \mathcal{V}(A)$, it is clear that E is homogeneous. Therefore, from (1.41) and (2.2), we have that Λ is also homogeneous. *QED*

Remark 2.4 That condition of linearity of the Jacobi structure does not necessarily imply that it is homogeneous is illustrated by the following simple example. Let M be a single point and $A^* = \mathbb{R}^2$ endowed with the Jacobi structure (Λ, E), where $\Lambda = xy\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$ and $E = x\frac{\partial}{\partial x}$. It is easy to prove that the Jacobi structure is linear. However, the Jacobi bracket of a linear function and the constant function 1 is not, in general, a basic function.

2.2 Homogeneous Jacobi structures and Jacobi algebroids

Let $\tau : A \to M$ be a vector bundle and A^* the dual bundle to A. Suppose that $\tau^* : A^* \to M$ is the canonical projection. It is well-known that there exists a one-to-one correspondence between Lie algebroid structures $([\![\ ,\ ]\!], \rho)$ on A and homogeneous (linear) Poisson structures on A^* (see [14, 15] and Section 1.2.2). Next, we will show an extension of the above results to the Jacobi setting.

Theorem 2.5 *Let $\tau : A \to M$ be a vector bundle over M and (Λ, E) be a homogeneous Jacobi structure on the dual bundle A^*. Then, (Λ, E) induces a Lie algebroid structure $([\![\ ,\ ]\!]^{(\Lambda,E)}, \rho^{(\Lambda,E)})$ on A and a 1-cocycle $\phi_0 \in \Gamma(A^*)$ for this structure characterized by the following relations*

$$\begin{aligned} &\widetilde{[\![X, Y]\!]}^{(\Lambda,E)} = \{\tilde{X}, \tilde{Y}\}_{(\Lambda,E)}, \\ &\rho^{(\Lambda,E)}(X)(f_M) \circ \tau^* = \{\tilde{X}, f_M \circ \tau^*\}_{(\Lambda,E)} - (f_M \circ \tau^*)\{\tilde{X}, 1\}_{(\Lambda,E)}, \\ &\phi_0(X) \circ \tau^* = \{\tilde{X}, 1\}_{(\Lambda,E)}, \end{aligned} \tag{2.3}$$

for $X, Y \in \Gamma(A)$ *and* $f_M \in C^\infty(M, \mathbb{R})$*, where* $\{\,,\,\}_{(\Lambda,E)}$ *is the Jacobi bracket associated with the Jacobi structure* (Λ, E)*.*

Proof: From Theorem 2.3, we have that $E \in \mathcal{V}(A)$ and there exists a linear Poisson structure Π_{A^*} on A^* such that

$$\Lambda = \Pi_{A^*} + E \wedge \Delta_{A^*}. \tag{2.4}$$

Using the results in Section 1.2.2 (see Example 8), we deduce that Π_{A^*} induces a Lie algebroid structure $([\![\,,\,]\!]^{\Pi_{A^*}}, \rho^{\Pi_{A^*}})$ on A and, from (1.39), (1.42) and (2.4), it follows that

$$\begin{aligned} &\widetilde{[\![X, Y]\!]}^{\Pi_{A^*}} = \{\tilde{X}, \tilde{Y}\}_{(\Lambda,E)}, \\ &\rho^{\Pi_{A^*}}(X)(f_M) \circ \tau^* = \{\tilde{X}, f_M \circ \tau^*\}_{(\Lambda,E)} - (f_M \circ \tau^*)\{\tilde{X}, 1\}_{(\Lambda,E)}, \end{aligned} \tag{2.5}$$

for $X, Y \in \Gamma(A)$ and $f_M \in C^\infty(M, \mathbb{R})$. Thus, the Lie algebroid structure $([\![\,,\,]\!]^{(\Lambda,E)}, \rho^{(\Lambda,E)})$ is just $([\![\,,\,]\!]^{\Pi_{A^*}}, \rho^{\Pi_{A^*}})$.

On the other hand, using Theorem 2.3, we have that $E \in \mathcal{V}(A)$ and, therefore, there exists a unique section ϕ_0 of A^* such that $E = -\phi_0^{\mathbf{v}}$. This implies that

$$\{\tilde{X}, 1\}_{(\Lambda,E)} = -E(\tilde{X}) = \phi_0(X) \circ \tau^*,$$

for $X \in \Gamma(A)$. Finally, from (2.5), we obtain that $\phi_0 \in \Gamma(A^*)$ is a 1-cocycle for the Lie algebroid $(A, [\![\,,\,]\!]^{(\Lambda,E)}, \rho^{(\Lambda,E)})$. *QED*

Motivated by Theorem 2.5, we introduce the following definition.

Definition 2.6 *A* Jacobi algebroid structure *on a vector bundle* $\tau : A \to M$ *is a pair* $(([\![\,,\,]\!], \rho), \phi_0)$*, where* $([\![\,,\,]\!], \rho)$ *is a Lie algebroid structure on* A *and* $\phi_0 \in \Gamma(A^*)$ *is a 1-cocycle.*

Now, we will prove a converse of Theorem 2.5.

Theorem 2.7 *Let* $\tau : A \to M$ *be a vector bundle and* $(([\![\,,\,]\!], \rho), \phi_0)$ *a Jacobi algebroid structure on* A*. Then, there is a unique Jacobi structure*

$(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$ *on* A^* *with Jacobi bracket which we will denote by* $\{,\}_{(A^*,\phi_0)}$ *satisfying*

$$\begin{aligned} \{\tilde{X},\tilde{Y}\}_{(A^*,\phi_0)} &= \widetilde{[\![X,Y]\!]}, \\ \{\tilde{X}, f_M\circ\tau^*\}_{(A^*,\phi_0)} &= (\rho(X)(f_M)+\phi_0(X)f_M)\circ\tau^*, \\ \{f_M\circ\tau^*, g_M\circ\tau^*\}_{(A^*,\phi_0)} &= 0, \end{aligned} \tag{2.6}$$

for $X, Y \in \Gamma(A)$ *and* $f_M, g_M \in C^\infty(M,\mathbb{R})$. *The Jacobi structure is homogeneous and it is given by*

$$\Lambda_{(A^*,\phi_0)} = \Pi_{A^*} + \Delta_{A^*}\wedge\phi_0^{\mathbf{v}}, \qquad E_{(A^*,\phi_0)} = -\phi_0^{\mathbf{v}}, \tag{2.7}$$

Π_{A^*} *being the linear Poisson structure on* A^* *induced by the Lie algebroid* $(A, [\![\,,\,]\!], \rho)$.

Proof: Denote by $\Lambda_{(A^*,\phi_0)}$ and $E_{(A^*,\phi_0)}$ the 2-vector and the vector field on A^* given by (2.7).

From (1.46), we obtain that $\phi_0^{\mathbf{v}}$ is an infinitesimal automorphism of the Poisson structure Π_{A^*}, that is,

$$(\mathcal{L}_0)_{\phi_0^{\mathbf{v}}}\Pi_{A^*} = 0. \tag{2.8}$$

Thus, using (1.41), (2.8) and since $\phi_0^{\mathbf{v}}$ is a homogeneous vector field, it follows that $(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$ is a homogeneous Jacobi structure on A^*. In addition, from (1.40), (1.42) and (2.7), we deduce that (2.6) holds.

Finally, it is clear that if (Λ, E) is a Jacobi structure on A^* which satisfies (2.6) then $\Lambda = \Lambda_{(A^*,\phi_0)}$ and $E = E_{(A^*,\phi_0)}$. *QED*

Remark 2.8 If $(([\![\,,\,]\!],\rho),\phi_0)$ is a Jacobi algebroid structure on a vector bundle $\tau : A \to M$ and (Λ, E) is a Jacobi structure on A^* such that

$$\{\tilde{X},\tilde{Y}\}_{(\Lambda,E)} = \widetilde{[\![X,Y]\!]}, \qquad \{\tilde{X},1\}_{(\Lambda,E)} = \phi_0(X)\circ\tau^*,$$

for $X, Y \in \Gamma(A)$, then, using Theorems 2.3 and 2.7, we deduce that $\Lambda = \Lambda_{(A^*,\phi_0)}$ and $E = E_{(A^*,\phi_0)}$.

Let M be a differentiable manifold and $\tau : A \to M$ be a vector bundle. Denote by $\mathcal{JA}$ and $\mathcal{HJ}$ the following sets: $\mathcal{JA}$ is the set of Jacobi algebroid structures on A and $\mathcal{HJ}$ is the set of the homogeneous Jacobi structures on A^*.

Then, using Theorems 2.5 and 2.7, we obtain

Theorem 2.9 *The mapping $\Psi : \mathcal{JA} \to \mathcal{HJ}$ between the sets $\mathcal{JA}$ and $\mathcal{HJ}$ given by*

$$\Psi(([\![\ ,\ ]\!], \rho), \phi_0) = (\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$$

is a bijection.

Note that $\Psi(\mathcal{LA}) = \mathcal{LP}$, where $\mathcal{LP}$ is the subset of the Jacobi structures of $\mathcal{HJ}$ which are Poisson and $\mathcal{LA}$ is the subset of $\mathcal{JA}$ of the pairs of the form $(([\![\ ,\ ]\!], \rho), 0)$, that is, $\mathcal{LP}$ is the set of linear Poisson structures on A^* and $\mathcal{LA}$ is the set of Lie algebroid structures on A. Therefore, from Theorem 2.9, we deduce a well known result (see [14, 15] and Example 8 in Section 1.2.2): the mapping Ψ induces a bijection between the sets $\mathcal{LA}$ and $\mathcal{LP}$.

2.3 Examples and applications

In this Section we will present some examples and applications of the results obtained in Section 2.2.

1.- Let $(\mathfrak{g}, [\ ,\ ]_{\mathfrak{g}})$ be a real Lie algebra of dimension n. The resultant Poisson structure $\Pi_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is the well known Lie-Poisson structure (see Examples 1.10). Thus, if $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle then, using Theorem 2.7, we deduce that

$$\Lambda_{(\mathfrak{g}^*,\phi_0)} = \Pi_{\mathfrak{g}^*} + \Delta_{\mathfrak{g}^*} \wedge C_{\phi_0},$$
$$E_{(\mathfrak{g}^*,\phi_0)} = -C_{\phi_0},$$

is a homogeneous Jacobi structure on $\mathfrak{g}^*$, where $\Delta_{\mathfrak{g}^*}$ is the radial (Liouville) vector field on $\mathfrak{g}^*$.

2.- Let $(TM, [\ ,\ ], Id)$ be the trivial Lie algebroid. In this case, the linear Poisson structure Π_{T^*M} on T^*M is the canonical symplectic structure. Therefore, if ϕ_0 is a closed 1-form on M, then the pair

$$\begin{aligned} \Lambda_{(T^*M,\phi_0)} &= \Pi_{T^*M} + \Delta_{T^*M} \wedge \phi_0^{\mathbf{v}}, \\ E_{(T^*M,\phi_0)} &= -\phi_0^{\mathbf{v}}, \end{aligned} \tag{2.9}$$

is a homogeneous Jacobi structure on T^*M. Furthermore, it is easy to prove that the map $\#_{\Lambda_{(T^*M,\phi_0)}} : \Omega^1(T^*M) \to \mathfrak{X}(T^*M)$ is an isomorphism of $C^\infty(M,\mathbb{R})$-modules. Therefore, $(T^*M, \Lambda_{(T^*M,\phi_0)}, E_{(T^*M,\phi_0)})$ is a transitive Jacobi manifold which implies that it is a l.c.s. manifold (see Remark 1.2). In fact, if λ_{T^*M} is the Liouville 1-form on T^*M and $\Omega_{T^*M} = -d_0\lambda_{T^*M}$ is the canonical symplectic 2-form then, using (1.12) and (2.9), we have that the l.c.s. structure $\Omega_{(T^*M,\phi_0)}$ and the Lee 1-form $\omega_{(T^*M,\phi_0)}$ on T^*M are given by

$$\begin{aligned} \Omega_{(T^*M,\phi_0)} &= \Omega_{T^*M} + \pi_M^*(\phi_0) \wedge \lambda_{T^*M}, \\ \omega_{(T^*M,\phi_0)} &= \pi_M^*(\phi_0), \end{aligned}$$

$\pi_M : T^*M \to M$ being the canonical projection. This l.c.s. structure was first considered in [40].

3.- Let (M,Π) be a Poisson manifold and $(T^*M, [\![\ ,\ ]\!]_\Pi, \#_\Pi)$ be the associated cotangent Lie algebroid. The induced Poisson structure on TM is the complete lift Π^c to TM of Π (see Examples 1.10). Thus, if $X \in \mathfrak{X}(M) = \Gamma(TM)$ is a 1-cocycle, that is, X is a Poisson infinitesimal automorphism of Π $((\mathcal{L}_0)_X\Pi = 0)$, we deduce that

$$\begin{aligned} \Lambda_{(TM,X)} &= \Pi^c + \Delta_{TM} \wedge X^{\mathbf{v}}, \\ E_{(TM,X)} &= -X^{\mathbf{v}}, \end{aligned}$$

is a homogeneous Jacobi structure on TM.

4.- The triple $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ is a Lie algebroid over M, where $\pi : TM \times \mathbb{R} \to TM$ is the canonical projection over the first factor and $[\ ,\ ]$ is the bracket given by (1.25). In this case, the linear Poisson structure $\Pi_{T^*M\times\mathbb{R}}$ on $T^*M \times \mathbb{R}$ is the *canonical cosymplectic structure* on $T^*M \times \mathbb{R}$, that is, $\Pi_{T^*M\times\mathbb{R}} = \Pi_{T^*M}$ and the pair $\phi_0 = (0,-1) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle of the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ (see

Example 3 in Section 1.2.2). Moreover, using Theorem 2.7, we have that the homogeneous Jacobi structure $(\Lambda_{(T^*M\times\mathbb{R},\phi_0)},\ E_{(T^*M\times\mathbb{R},\phi_0)})$ on $T^*M\times\mathbb{R}$ is the one defined by the *canonical contact 1-form* η_M. We recall that if $\pi_1 : T^*M\times\mathbb{R}\to T^*M$ and $\pi_2 : T^*M\times\mathbb{R}\to\mathbb{R}$ are the canonical projections then η_M is the 1-form on $T^*M\times\mathbb{R}$ given by (see [72])

$$\eta_M = \pi_2^*(d_0t) - \pi_1^*(\lambda_{T^*M}) \tag{2.10}$$

and that the local expressions of the Poisson structure Π_{T^*M} and the Jacobi structure associated with η_M are

$$\begin{aligned} \Pi_{T^*M\times\mathbb{R}} &= \sum_{i=1}^m \frac{\partial}{\partial q^i}\wedge\frac{\partial}{\partial p_i},\\ \Lambda_{(T^*M\times\mathbb{R},\phi_0)} &= \sum_{i=1}^m \Big(\frac{\partial}{\partial q^i}+p_i\frac{\partial}{\partial t}\Big)\wedge\frac{\partial}{\partial p_i},\quad E_{(T^*M\times\mathbb{R},\phi_0)} = \frac{\partial}{\partial t}, \end{aligned}\tag{2.11}$$

$(q^1,\dots,q^m,p_1,\dots,p_m,t)$ being fibred coordinates on $T^*M\times\mathbb{R}$.

5.- Let (M,Λ,E) be a Jacobi manifold. Then, the vector bundle $T^*M\times\mathbb{R}\to M$ admits a Lie algebroid structure $([\![\,,\,]\!]_{(\Lambda,E)},\widetilde{\#}_{(\Lambda,E)})$ and the pair $\phi_0=(-E,0)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\cong\Gamma(TM\times\mathbb{R})$ is a 1-cocycle for this Lie algebroid (see Example 4 in Section 1.2.2). Moreover, using Examples 1.10 and Theorem 2.7, we deduce that the homogeneous Jacobi structure $(\Lambda_{(TM\times\mathbb{R},\phi_0)},E_{(TM\times\mathbb{R},\phi_0)})$ on $TM\times\mathbb{R}$ is given by

$$\begin{aligned} \Lambda_{(TM\times\mathbb{R},\phi_0)} &= \Lambda^c + \frac{\partial}{\partial t}\wedge E^c - t\Big(\Lambda^{\mathbf{v}} + \frac{\partial}{\partial t}\wedge E^{\mathbf{v}}\Big),\\ E_{(TM\times\mathbb{R},\phi_0)} &= E^{\mathbf{v}}, \end{aligned}$$

where Λ^c and E^c (resp. $\Lambda^{\mathbf{v}}$ and $E^{\mathbf{v}}$) is the complete (resp. vertical) lift to TM of Λ and E, respectively. We remark that in [43] the authors characterize the conformal infinitesimal automorphisms of (M,Λ,E) as Legendre-Lagrangian submanifolds of the Jacobi manifold $(TM\times\mathbb{R},\Lambda_{(TM\times\mathbb{R},\phi_0)},E_{(TM\times\mathbb{R},\phi_0)})$.

6.- Let $(([\![\,,\,]\!],\rho),\phi_0)$ be a Jacobi algebroid structure on a vector bundle $A\to M$. Denote by $\bar{A}$ the product $A\times\mathbb{R}$ and by $\Pi_{\bar{A}^*}$ the Poissonization of the homogeneous Jacobi structure $(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$ (see (1.19)). From (1.19) and Theorem 2.7, it follows that $\Pi_{\bar{A}^*}$ is a linear Poisson structure on the vector bundle $\bar{A}^* = A^* \times \mathbb{R} \to M \times \mathbb{R}$. Thus, the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ admits a Lie algebroid structure $([\![\, , \,]\!]^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$. Note that the space $\Gamma(\bar{A})$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification and using (1.2), (1.19), (1.42) and Theorem 2.7, we deduce that

$$
\begin{aligned}
[\![\bar{X}, \bar{Y}]\!]^{\hat{}\phi_0} &= e^{-t}\Big([\![\bar{X}, \bar{Y}]\!] + \phi_0(\bar{X})(\frac{\partial \bar{Y}}{\partial t} - \bar{Y}) - \phi_0(\bar{Y})(\frac{\partial \bar{X}}{\partial t} - \bar{X})\Big), \\
\hat{\rho}^{\phi_0}(\bar{X}) &= e^{-t}\Big(\rho(\bar{X}) + \phi_0(\bar{X})\frac{\partial}{\partial t}\Big),
\end{aligned}
\tag{2.12}
$$

for all $\bar{X}, \bar{Y} \in \Gamma(\bar{A})$, where $\dfrac{\partial \bar{X}}{\partial t}$ (resp., $\dfrac{\partial \bar{Y}}{\partial t}$) is the derivative of $\bar{X}$ (resp., $\bar{Y}$) with respect to the time. Note that if $t \in \mathbb{R}$ then the sections $\bar{X}$ and $\bar{Y}$ induce, in a natural way, two sections $\bar{X}_t$ and $\bar{Y}_t$ of $A \to M$ and that $[\![\bar{X}, \bar{Y}]\!]$ and $\rho(\bar{X})$ are the time-dependent sections of $A \to M$ given by $[\![\bar{X}, \bar{Y}]\!](x,t) = [\![\bar{X}_t, \bar{Y}_t]\!](x)$ and the vector field $\rho(\bar{X})$ on $M \times \mathbb{R}$ defined by $\rho(\bar{X})(x,t) = \rho(\bar{X}_t)(x)$, for all $(x,t) \in M \times \mathbb{R}$.

2.4 The characteristic foliation of a homogeneous Jacobi structure on a vector space

Let $\mathfrak{g}$ be a real vector space of finite dimension and $\Pi_{\mathfrak{g}^*}$ be a linear Poisson structure on $\mathfrak{g}^*$. The Poisson structure $\Pi_{\mathfrak{g}^*}$ induces a Lie algebra structure on $\mathfrak{g}$. Denote by G a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then, the leaves of the symplectic foliation associated with $\Pi_{\mathfrak{g}^*}$ are the orbits of the coadjoint representation associated with G. In this Section we will obtain the corresponding result in the Jacobi setting.

First of all, we must replace the terms linear and Poisson by the terms homogeneous and Jacobi, respectively. So, suppose that (Λ, E) is a homogeneous Jacobi structure on the dual vector space $\mathfrak{g}^*$ of a real vector space $\mathfrak{g}$. Then, from Theorems 2.5 and 2.7 (see also Example 1 in Section 2.3), we deduce that there exists a Lie algebra structure $[\, ,\,]_{\mathfrak{g}}$ on $\mathfrak{g}$ and a 1-cocycle $\phi_0 \in \mathfrak{g}^*$ of $(\mathfrak{g}, [\, ,\,]_{\mathfrak{g}})$ such that

$$\Lambda = \Pi_{\mathfrak{g}^*} + \Delta_{\mathfrak{g}^*} \wedge C_{\phi_0}, \qquad E = -C_{\phi_0}. \tag{2.13}$$

Moreover, if $f \in C^\infty(\mathfrak{g}^*, \mathbb{R})$ and $\mathcal{H}_f^{(\Lambda,E)}$ is the hamiltonian vector field of f with respect to (Λ, E) then the mapping

$$-\mathcal{H}^{(\Lambda,E)} : \mathfrak{g} \to \mathfrak{X}(\mathfrak{g}^*), \qquad Y \mapsto -\mathcal{H}^{(\Lambda,E)}(Y) = -\mathcal{H}_{\tilde{Y}}^{(\Lambda,E)}$$

is a Lie algebra anti-homomorphism (see (1.16) and (2.13)). However, $E \notin -\mathcal{H}^{(\Lambda,E)}(\mathfrak{g})$ because the constant function 1 is not a linear function on $\mathfrak{g}^*$.

The way of solving this problem is the following one. Consider the semi-direct Lie algebra structure on $\tilde{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$ given by

$$[(Y, \lambda), (Z, \gamma)]_{\tilde{\mathfrak{g}}} = ([Y, Z]_{\mathfrak{g}}, \gamma\, \phi_0(Y) - \lambda\, \phi_0(Z)),$$

for $(Y, \lambda), (Z, \gamma) \in \tilde{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}$. Now, using (1.16) and (2.13), we obtain that the mapping

$$\Phi : \tilde{\mathfrak{g}} \to \mathfrak{X}(\mathfrak{g}^*), \quad (Y, \lambda) \mapsto \Phi(Y, \lambda) = -\mathcal{H}_{\tilde{Y}}^{(\Lambda,E)} - \lambda\, E,$$

is a Lie algebra anti-homomorphism. Thus, under the canonical identification $T_\mu \mathfrak{g}^* \cong \mathfrak{g}^*$, for all $\mu \in \mathfrak{g}^*$, it follows that Φ defines a linear representation of $\tilde{\mathfrak{g}}$ on $\mathfrak{g}^*$ which we will also denote by $\Phi : \tilde{\mathfrak{g}} \times \mathfrak{g}^* \to \mathfrak{g}^*$. In fact, using (1.15) and (2.13), we deduce that

$$\Phi((Y, \lambda), \mu) = \mathrm{coad}_Y^{\mathfrak{g}} \mu + \phi_0(Y)\mu + \lambda\, \phi_0,$$

for $(Y, \lambda) \in \tilde{\mathfrak{g}}$ and $\mu \in \mathfrak{g}^*$, where $\mathrm{coad}^{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation associated with $\mathfrak{g}$.

On the other hand, if $\mathcal{F}^{(\Lambda,E)}$ is the characteristic foliation on $\mathfrak{g}^*$ associated with the Jacobi structure (Λ, E), it is clear that

$$\mathcal{F}_\mu^{(\Lambda,E)} = \{\Phi((Y, \lambda), \mu) \in \mathfrak{g}^* \cong T_\mu \mathfrak{g}^* \,/\, (Y, \lambda) \in \tilde{\mathfrak{g}} = \mathfrak{g} \times \mathbb{R}\}.$$

Next, we consider a connected and simply connected Lie group $\tilde{G}$ with Lie algebra $(\tilde{\mathfrak{g}}, [\,,\,]_{\tilde{\mathfrak{g}}})$. Since $\Phi : \tilde{\mathfrak{g}} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is a linear representation, there exists a linear representation of $\tilde{G}$ on $\mathfrak{g}^*$

$$\overline{\mathrm{Coad}} : \tilde{G} \times \mathfrak{g}^* \to \mathfrak{g}^*,$$

such that the associated linear representation of $\tilde{\mathfrak{g}}$ on $\mathfrak{g}^*$, $\overline{\mathrm{coad}} : \tilde{\mathfrak{g}} \times \mathfrak{g}^* \to \mathfrak{g}^*$, is just Φ. Consequently,

Theorem 2.10 *Let $\mathfrak{g}$ be a real vector space of finite dimension and (Λ, E) be a homogeneous Jacobi structure over $\mathfrak{g}^*$. Then, the leaves of the characteristic foliation associated with the Jacobi structure (Λ, E) are just the orbits of the linear representation $\overline{Coad} : \tilde{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$.*

In the following, we will give an explicit description of the Lie group $\tilde{G}$ and the linear representation $\overline{\mathrm{Coad}}$.

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Since ϕ_0 is a 1-cocycle, then there exists a unique multiplicative function $\sigma_0 : G \to \mathbb{R}$ such that

$$(d_0\sigma_0)(\mathfrak{e}) = \phi_0. \tag{2.14}$$

We recall that $\sigma_0 : G \to \mathbb{R}$ is multiplicative if $\sigma_0(gh) = \sigma_0(g) + \sigma_0(h)$, for $g, h \in G$.

Thus, using the results in [112], the Lie group $\tilde{G}$ is isomorphic to the product $G \times \mathbb{R}$ and the multiplication in $\tilde{G} = G \times \mathbb{R}$ is given by

$$(g_1, t_1)(g_2, t_2) = (g_1 g_2, t_1 + e^{\sigma_0(g_1)} t_2), \tag{2.15}$$

for all $(g_1, t_1), (g_2, t_2) \in \tilde{G} = G \times \mathbb{R}$, that is, $\tilde{G}$ is the semi-direct product Lie group $G \times_{\psi_{\sigma_0}} \mathbb{R}$, associated with the linear representation $\psi_{\sigma_0} : G \times \mathbb{R} \to \mathbb{R}$, $(g, t) \mapsto t\, e^{\sigma_0(g)}$.

Next, we will describe the linear representation $\overline{\mathrm{Coad}} : \tilde{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$.

Theorem 2.11 *Let $\mathfrak{g}$ be a real vector space of finite dimension and (Λ, E) be a homogeneous Jacobi structure over $\mathfrak{g}^*$. Then, the leaves of the characteristic* *foliation associated with* (Λ, E) *are the orbits of the linear representation* $\overline{Coad} : \tilde{G} \times \mathfrak{g}^* \to \mathfrak{g}^*$ *given by*

$$\overline{Coad}_{(g,t)}(\mu) = e^{\sigma_0(g)}\, Coad^G_g \mu + t\phi_0, \tag{2.16}$$

for $(g,t) \in \tilde{G} = G \times \mathbb{R}$ *and* $\mu \in \mathfrak{g}^*$*, where* $Coad^G : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ *is the coadjoint representation associated with* G*.*

Proof: Using (2.15), we obtain that the coadjoint representation associated with $\tilde{G}$, $\mathrm{Coad}^{\tilde{G}} : \tilde{G} \times \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$, is given by

$$\mathrm{Coad}^{\tilde{G}}_{(g,t)}(\mu, \gamma) = (\mathrm{Coad}^G_g \mu + \gamma t e^{-\sigma_0(g)}\phi_0, \gamma e^{-\sigma_0(g)}), \tag{2.17}$$

for all $(g,t) \in \tilde{G} = G \times \mathbb{R}$ and $(\mu, \gamma) \in \tilde{\mathfrak{g}}^* = \mathfrak{g}^* \times \mathbb{R}$.

From (2.17), it follows that the action $\widetilde{\mathrm{Coad}} : \tilde{G} \times \tilde{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$ of $\tilde{G}$ on $\tilde{\mathfrak{g}}^*$ defined by

$$\widetilde{\mathrm{Coad}}_{(g,t)}(\mu, \gamma) = e^{\sigma_0(g)} \mathrm{Coad}^{\tilde{G}}_{(g,t)}(\mu, \gamma) \tag{2.18}$$

satisfies

$$\widetilde{\mathrm{Coad}}_{(g,t)}(\mu, 1) \in \mathfrak{g}^* \times \{1\},$$

for all $(g,t) \in \tilde{G} \cong G \times \mathbb{R}$ and $\mu \in \mathfrak{g}^*$. In addition, the restriction to $\mathfrak{g}^* \times \{1\} \cong \mathfrak{g}^*$ of the infinitesimal generator of $(Y, \lambda) \in \tilde{\mathfrak{g}} \cong \mathfrak{g} \times \mathbb{R}$ with respect to $\widetilde{\mathrm{Coad}}$, $(Y,\lambda)^{\widetilde{\mathrm{Coad}}}_{\tilde{\mathfrak{g}}^*}$, is just $\Phi(Y, \lambda)$, that is,

$$((Y,\lambda)^{\widetilde{\mathrm{Coad}}}_{\tilde{\mathfrak{g}}^*})_{|\mathfrak{g}^*} = \Phi(Y, \lambda).$$

Consequently, the restriction to $\mathfrak{g}^* \times \{1\} \cong \mathfrak{g}^*$ of $\widetilde{\mathrm{Coad}}_{(g,t)}$ is just $\overline{\mathrm{Coad}}_{(g,t)}$, for all $(g,t) \in \tilde{G}$. Finally, using (2.17) and (2.18), we obtain (2.16). *QED*

Theorem 2.11 allows us to describe the Jacobi structure on the leaves of the characteristic foliation of a homogeneous Jacobi structure on a vector space.

Theorem 2.12 *Let* $\mathfrak{g}$ *be a real vector space of finite dimension and* (Λ, E) *be a homogeneous Jacobi structure on* $\mathfrak{g}^*$*. Consider* $\mu \in \mathfrak{g}^*$ *and* L_μ *the leaf of the characteristic foliation over the point* μ *associated with* (Λ, E)*.*

i) If $E(\mu) \notin \#_\Lambda(T^*_\mu\mathfrak{g}^*)$ *and* $\nu \in L_\mu$ *then*

$$T_\nu L_\mu =< \{Y_\nu = Y_{\mathfrak{g}^*}^{Coad^G}(\nu) + \phi_0(Y)\Delta_{\mathfrak{g}^*}(\nu)\}_{Y\in\mathfrak{g}}, \phi_0^{\mathbf{v}}(\nu) >$$

and (Λ, E) *induces a contact structure* η^{L_μ} *on* L_μ *defined by*

$$\eta^{L_\mu}(\nu)(Y_\nu) = -\nu(Y), \qquad \eta^{L_\mu}(\nu)(\phi_0^{\mathbf{v}}(\nu)) = -1,$$

for all $Y \in \mathfrak{g}$.

ii) If $E(\mu) \in \#_\Lambda(T^*_\mu\mathfrak{g}^*)$ *and* $\nu \in L_\mu$ *then*

$$T_\nu L_\mu =< \{Y_\nu = Y_{\mathfrak{g}^*}^{Coad^G}(\nu) + \phi_0(Y)\Delta_{\mathfrak{g}^*}(\nu)\}_{Y\in\mathfrak{g}} >$$

and (Λ, E) *induces a l.c.s. structure* $(\Omega^{L_\mu}, \omega^{L_\mu})$ *on* L_μ *defined by*

$$\begin{aligned} \Omega^{L_\mu}(\nu)(Y_\nu, Z_\nu) &= \nu([Y,Z]_\mathfrak{g}), \\ \omega^{L_\mu}(\nu)(Y_\nu) &= -\phi_0(Y), \end{aligned}$$

for all $Y, Z \in \mathfrak{g}$.

Proof: If $(Y,\lambda) \in \tilde{\mathfrak{g}} \cong \mathfrak{g} \times \mathbb{R}$ and $\nu \in \mathfrak{g}^*$ is a point of L_μ then

$$\begin{aligned} \mathcal{H}^{(\Lambda,E)}_{(\lambda+\tilde{Y})}(\nu) &= -\overline{\mathrm{coad}}((Y,\lambda),\nu) \\ &= -Y_{\mathfrak{g}^*}^{\mathrm{Coad}^G}(\nu) - \phi_0(Y)\Delta_{\mathfrak{g}^*}(\nu) - \lambda C_{\phi_0}(\nu), \end{aligned}$$

where $\lambda+\tilde{Y}$ is the function on $\mathfrak{g}^*$ given by $(\lambda+\tilde{Y})(\nu) = \lambda + \nu(Y)$, for $\nu \in \mathfrak{g}^*$. Thus, using (1.17) and (1.18), we deduce the result. *QED*

CHAPTER 3

Jacobi structures and Jacobi bialgebroids

In this Chapter, we introduce the notion of a Jacobi bialgebroid (a generalization of the notion of a Lie bialgebroid) in such a way that a Jacobi manifold has associated a canonical Jacobi bialgebroid. Furthermore, some properties of Jacobi algebroids are proved, the relation with Lie bialgebroids is discussed and several examples of Jacobi bialgebroids are given. In the last part of the Chapter, a characterization of Jacobi bialgebroids in terms of Jacobi algebroid morphisms is obtained.

3.1 Differential calculus on Jacobi algebroids

In this Section, we will develop a differential calculus for Jacobi algebroids.

3.1.1 ϕ_0-differential and ϕ_0-Lie derivative

Let $((\llbracket\, , \,\rrbracket, \rho), \phi_0)$ be a Jacobi algebroid structure on a vector bundle $\tau : A \to M$. Using (1.22), we can define a representation ρ^{ϕ_0} of the Lie algebroid $(A, \llbracket\, , \,\rrbracket, \rho)$ on the trivial vector bundle $M \times \mathbb{R} \to M$ given by

$$\rho^{\phi_0}(X)f = \rho(X)(f) + \phi_0(X)f, \tag{3.1}$$

for $X \in \Gamma(A)$ and $f \in C^\infty(M,\mathbb{R})$. Thus, one can consider the standard cohomology complex associated with the vector bundle $M \times \mathbb{R} \to M$ and the representation ρ^{ϕ_0} (see [82]). The cohomology operator $d^{\phi_0} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^{k+1} A^*)$ of this complex will be called the ϕ_0-*differential* of A. If d is the differential of the Lie algebroid $(A, [\![\,,\,]\!], \rho)$ then we have that

$$d^{\phi_0}\mu = d\mu + \phi_0 \wedge \mu, \tag{3.2}$$

for $\mu \in \Gamma(\wedge^k A^*)$.

Remark 3.1 If ϕ_0 is a closed 1-form on a manifold M then ϕ_0 is a 1-cocycle for the trivial Lie algebroid $(TM, [\,,\,], Id)$ and we can consider the operator d^{ϕ_0}. Some results about the cohomology defined by d^{ϕ_0} were obtained in [39, 66, 108]. These results were used in the study of locally conformal Kähler and locally conformal symplectic structures.

If $k \geq 0$ and $X \in \Gamma(A)$, we can also define the Lie derivative (associated with the representation ρ^{ϕ_0}) with respect to X, $\mathcal{L}_X^{\phi_0} : \Gamma(\wedge^k A^*) \to \Gamma(\wedge^k A^*)$, as follows (see [82]):

$$\mathcal{L}_X^{\phi_0} = d^{\phi_0} \circ i_X + i_X \circ d^{\phi_0}. \tag{3.3}$$

It is called the ϕ_0-*Lie derivative* with respect to X. A direct computation proves that if $\mathcal{L}$ is the usual Lie derivative of the Lie algebroid $(A, [\![\,,\,]\!], \rho)$ then

$$\mathcal{L}_X^{\phi_0}\mu = \mathcal{L}_X\mu + \phi_0(X)\mu, \tag{3.4}$$

for any $\mu \in \Gamma(\wedge^k A^*)$.

Remark 3.2 *i)* If we consider the Lie algebroid $(TM \times \mathbb{R}, [\;,\;], \pi)$ then $\phi_0 = (0,1) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle (see Example 3 in Section 1.2.2). Thus, we have the corresponding representation $\pi^{(0,1)}$ of $TM \times \mathbb{R}$ on the vector bundle $M \times \mathbb{R} \to M$ which, in this case, is defined by

$$\pi^{(0,1)}((X,f),g) = X(g) + fg, \tag{3.5}$$

for $(X,f) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R})$ and $g \in C^\infty(M,\mathbb{R})$. From (1.24), (1.26) and (3.2), we obtain that the ϕ_0-differential $\tilde{d}_0^{(0,1)}$ is given by

$$\tilde{d}_0^{(0,1)}(\mu,\nu) = (d_0\mu, \mu - d_0\nu), \tag{3.6}$$

for $(\mu,\nu)\in\Omega^k(M)\oplus\Omega^{k-1}(M)\cong\Gamma(\wedge^k(T^*M\times\mathbb{R}))$. Moreover, using (1.24), (3.3) and (3.6), we deduce that

$$(\tilde{\mathcal{L}}_0^{(0,1)})_{(X,f)}(\mu,\nu)=((\mathcal{L}_0)_X\mu+d_0f\wedge\nu+f\,\mu,(\mathcal{L}_0)_X\nu+f\,\nu), \tag{3.7}$$

for $(X,f)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\cong\Gamma(TM\times\mathbb{R})$, where $\tilde{\mathcal{L}}_0^{(0,1)}$ is the ϕ_0-Lie derivative of $(TM\times\mathbb{R},([\ ,\ ],\pi),(0,1))$.

ii) Assume that (Λ,E) is a Jacobi structure on M. Then, we may consider the 1-jet Lie algebroid $(T^*M\times\mathbb{R},[\![\ ,\ ]\!]_{(\Lambda,E)},\widetilde{\#}_{(\Lambda,E)})$ and the corresponding homomorphism of $C^\infty(M,\mathbb{R})$-modules $\#_{(\Lambda,E)}:\Gamma(T^*M\times\mathbb{R})\cong\Omega^1(M)\times C^\infty(M,\mathbb{R})\to\Gamma(TM\times\mathbb{R})\cong\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$. Furthermore, a long computation, using (1.23), (1.24), (3.6) and (3.7), shows that the Lie algebroid bracket $[\![\ ,\ ]\!]_{(\Lambda,E)}$ and the anchor map $\widetilde{\#}_{(\Lambda,E)}$ can be written in terms of the homomorphism $\#_{(\Lambda,E)}:\Gamma(T^*M\times\mathbb{R})\to\Gamma(TM\times\mathbb{R})$ and the operators $\tilde{\mathcal{L}}_0^{(0,1)}$ and $\tilde{d}_0^{(0,1)}$ as follows

$$\begin{aligned}
[\![(\mu,f),(\nu,g)]\!]_{(\Lambda,E)} &= (\tilde{\mathcal{L}}_0^{(0,1)})_{\#_{(\Lambda,E)}(\mu,f)}(\nu,g)-(\tilde{\mathcal{L}}_0^{(0,1)})_{\#_{(\Lambda,E)}(\nu,g)}(\mu,f)\\
&\quad -\tilde{d}_0^{(0,1)}\Big((\Lambda,E)((\mu,f),(\nu,g))\Big)\\
&= i_{\#_{(\Lambda,E)}(\mu,f)}(\tilde{d}_0^{(0,1)}(\nu,g))-i_{\#_{(\Lambda,E)}(\nu,g)}(\tilde{d}_0^{(0,1)}(\mu,f))\\
&\quad +\tilde{d}_0^{(0,1)}\Big((\Lambda,E)((\mu,f),(\nu,g))\Big),\\
\widetilde{\#}_{(\Lambda,E)} &= \pi\circ\#_{(\Lambda,E)}.
\end{aligned} \tag{3.8}$$

Compare equation (1.31) with the above expression of the Lie algebroid bracket $[\![\ ,\ ]\!]_{(\Lambda,E)}$.

iii) Let $(A,([\![\ ,\ ]\!],\rho),\phi_0)$ be a Jacobi algebroid over M. The homomorphism of $C^\infty(M,\mathbb{R})$-modules $(\rho,\phi_0):\Gamma(A)\to\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$ given by

$$X\mapsto(\rho(X),\phi_0(X)), \tag{3.9}$$

induces a Lie algebroid homomorphism over the identity between the Lie algebroids $(A,[\![\ ,\ ]\!],\rho)$ and $(TM\times\mathbb{R},[\ ,\ ],\pi)$, that is, $\pi\circ(\rho,\phi_0)=\rho$ and

$$(\rho,\phi_0)[\![X,Y]\!]=[(\rho,\phi_0)(X),(\rho,\phi_0)(Y)], \tag{3.10}$$

for $X, Y \in \Gamma(A)$. Moreover, if $(\rho,\phi_0)^* : \Omega^1(M)\times C^\infty(M,\mathbb{R}) \to \Gamma(A^*)$ is the adjoint homomorphism of (ρ,ϕ_0), then

$$(\rho,\phi_0)^*(0,1) = \phi_0.$$

As a consequence,

$$\begin{aligned}(\rho,\phi_0)^*(\tilde{d}_0^{(0,1)} f) &= (\rho,\phi_0)^*(d_0 f, f) = d^{\phi_0} f,\\ (\rho,\phi_0)^*(d_0 f, 0) &= df,\end{aligned} \tag{3.11}$$

for $f \in C^\infty(M,\mathbb{R})$.

3.1.2 ϕ_0-Schouten bracket

In [3], a skew-symmetric Schouten bracket was defined for two multilinear maps of a commutative associative algebra $\mathfrak{F}$ over $\mathbb{R}$ with unit as follows. Let $\mathcal{P}$ and $\mathcal{P}'$ be skew-symmetric multilinear maps of degree k and k', respectively, and $f_1,\dots,f_{k+k'-1} \in \mathfrak{F}$. If A is any subset of $\{1,2,\dots,(k+k'-1)\}$, let A' denote its complement and $|A|$ the number of elements in A. If $|A| = l$ and the elements in A are $\{i_1,\dots,i_l\}$ in increasing order, let us write f_A for the ordered k-uple $(f_{i_1},\dots,f_{i_l})$. Furthermore, we write ε_A for the sign of the permutation which rearranges the elements of the ordered $(k+k'-1)$-uple (A',A), in the original order. Then, the Schouten bracket of $\mathcal{P}$ and $\mathcal{P}'$, $[\mathcal{P},\mathcal{P}']^{(0,1)}$, is the skew-symmetric multilinear map of degree $k+k'-1$ given by

$$\begin{aligned}&[\mathcal{P},\mathcal{P}']^{(0,1)}(f_1,\dots f_{k+k'-1})\\ &\quad = \sum_{|A|=k'} \varepsilon_A \mathcal{P}(\mathcal{P}'(f_A), f_{A'}) + (-1)^{kk'} \sum_{|B|=k} \varepsilon_B \mathcal{P}'(\mathcal{P}(f_B), f_{B'}).\end{aligned}$$

One can prove that if $\mathcal{P}$ and $\mathcal{P}'$ are first-order differential operators on each of its arguments, so is $[\mathcal{P},\mathcal{P}']^{(0,1)}$. In particular, if M is a differentiable manifold and $\mathfrak{F} = C^\infty(M,\mathbb{R})$, we know that a k-linear skew-symmetric first-order differential operator can be identified with a pair $(P,Q) \in \mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$

(that is, a k-section of $TM \times \mathbb{R} \to M$) in such a way that

$$(P,Q)(f_1,\ldots,f_k) = P(d_0 f_1,\ldots,d_0 f_k) + \sum_{i=1}^{k}(-1)^{i+1} f_i\, Q(d_0 f_1,\ldots,\widehat{d_0 f_i},\ldots,d_0 f_k),$$

for $f_1,\ldots,f_k \in C^\infty(M,\mathbb{R})$. Under the above identification, we have that

$$\begin{aligned}[(P,Q),(P',Q')]^{(0,1)} = \Big(&[P,P'] + (-1)^{k+1}(k-1)P\wedge Q' - (k'-1)Q\wedge P', \\ &(-1)^{k+1}[P,Q'] - [Q,P'] + (-1)^{k+1}(k-k')Q\wedge Q'\Big),\end{aligned} \tag{3.12}$$

for $(P,Q) \in \mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$ and $(P',Q')\in\mathcal{V}^{k'}(M)\oplus\mathcal{V}^{k'-1}(M)$. If $[\,,\,]$ is the Schouten bracket of the Lie algebroid $(TM\times\mathbb{R},[\,,\,],\pi)$, an easy computation, using (1.24), (1.27) and (3.12), shows that

$$\begin{aligned}[(P,Q),(P',Q')]^{(0,1)} = {}&[(P,Q),(P',Q')] + (-1)^{k+1}(k-1)(P,Q)\wedge \\ &(i_{(0,1)}(P',Q')) - (k'-1)(i_{(0,1)}(P,Q))\wedge(P',Q').\end{aligned} \tag{3.13}$$

Remark 3.3 *i)* Note that a 2-section of the vector bundle $TM\times\mathbb{R}\to M$ defines a Jacobi structure on M if and only if

$$[(\Lambda,E),(\Lambda,E)]^{(0,1)} = 0 \tag{3.14}$$

(see (1.1) and (3.12)).

ii) Suppose that (Λ,E) is a Jacobi structure on M. Then, the vector bundle $T^*M\times\mathbb{R}\to M$ is a Lie algebroid and $X_0 = (-E,0)\in\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\cong\Gamma(TM\times\mathbb{R})$ is a 1-cocycle of this Lie algebroid (see Example 4 in Section 1.2.2). Moreover, if d_* is the differential of $T^*M\times\mathbb{R}$, using (1.30) and (3.12), we have that the X_0-differential $d_*^{X_0} = d_*^{(-E,0)}$ is given by

$$d_*^{(-E,0)}(P,Q) = -[(\Lambda,E),(P,Q)]^{(0,1)}, \tag{3.15}$$

for $(P,Q)\in\mathcal{V}^k(M)\oplus\mathcal{V}^{k-1}(M)$. Compare equation (3.15) with the expression of the differential of the Lie algebroid associated with a Poisson manifold (see (1.32)).

Suggested by (3.13), we prove the following result.

Theorem 3.4 *Let* $(A, ([\![\,,\,]\!], \rho), \phi_0)$ *be a Jacobi algebroid over* M. *Then, there exists a unique operation* $[\![\,,\,]\!]^{\phi_0} : \Gamma(\wedge^* A) \times \Gamma(\wedge^* A) \to \Gamma(\wedge^* A)$ *such that*

$$[\![P, P']\!]^{\phi_0} \in \Gamma(\wedge^{k+k'-1} A), \tag{3.16}$$

$$[\![X, f]\!]^{\phi_0} = \rho^{\phi_0}(X)(f), \tag{3.17}$$

$$[\![X, Y]\!]^{\phi_0} = [\![X, Y]\!], \tag{3.18}$$

$$[\![P, P']\!]^{\phi_0} = (-1)^{kk'} [\![P', P]\!]^{\phi_0}, \tag{3.19}$$

$$\begin{aligned} [\![P, P' \wedge P'']\!]^{\phi_0} \;=\;& [\![P, P']\!]^{\phi_0} \wedge P'' + (-1)^{k'(k+1)} P' \wedge [\![P, P'']\!]^{\phi_0} \\ & -(i_{\phi_0} P) \wedge P' \wedge P'', \end{aligned} \tag{3.20}$$

for $f \in C^\infty(M, \mathbb{R})$, $X, Y \in \Gamma(A)$, $P \in \Gamma(\wedge^k A)$, $P' \in \Gamma(\wedge^{k'} A)$ *and* $P'' \in \Gamma(\wedge^{k''} A)$. *This operation is given by the general formula*

$$[\![P, P']\!]^{\phi_0} = [\![P, P']\!] + (-1)^{k+1}(k-1) P \wedge (i_{\phi_0} P') - (k'-1)(i_{\phi_0} P) \wedge P'.$$

Furthermore, it satisfies the graded Jacobi identity

$$\begin{aligned} &(-1)^{kk''} [\![[\![P, P']\!]^{\phi_0}, P'']\!]^{\phi_0} + (-1)^{k'k''} [\![[\![P'', P]\!]^{\phi_0}, P']\!]^{\phi_0} \\ &\quad + (-1)^{kk'} [\![[\![P', P'']\!]^{\phi_0}, P]\!]^{\phi_0} = 0. \end{aligned} \tag{3.21}$$

Proof: We define the operation $[\![\,,\,]\!]^{\phi_0} : \Gamma(\wedge^* A) \times \Gamma(\wedge^* A) \to \Gamma(\wedge^* A)$ by

$$[\![P, P']\!]^{\phi_0} = [\![P, P']\!] + (-1)^{k+1}(k-1) P \wedge (i_{\phi_0} P') - (k'-1)(i_{\phi_0} P) \wedge P', \tag{3.22}$$

for $P \in \Gamma(\wedge^k A)$ and $P' \in \Gamma(\wedge^{k'} A)$. Using (3.22) and the properties of the Schouten bracket of multi-sections of A, we deduce (3.16), (3.17), (3.18), (3.19) and (3.20).

To prove the graded Jacobi identitity, we proceed as follows. If d is the differential of the Lie algebroid A and $\mu \in \Gamma(A^*)$ is a 1-cocycle, we have that

$$i_\mu [\![X, P']\!] = [\![X, i_\mu P']\!] - i_{d(\mu(X))} P',$$

for $X \in \Gamma(A)$ and $P' \in \Gamma(\wedge^{k'} A)$. Using this relation and the fact that

$$[\![X_1 \wedge \ldots \wedge X_k, P']\!] = \sum_{i=1}^{k} (-1)^{i+1} X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_k \wedge [\![X_i, P']\!],$$

for $X_1, \ldots, X_k \in \Gamma(A)$, it follows that

$$i_\mu [\![P, P']\!] = -[\![i_\mu P, P']\!] + (-1)^{k+1} [\![P, i_\mu P']\!], \tag{3.23}$$

for $P \in \Gamma(\wedge^k A)$. From (3.22) and (3.23), we deduce that

$$i_{\phi_0}([\![P, P']\!]^{\phi_0}) = -[\![i_{\phi_0} P, P']\!]^{\phi_0} + (-1)^{k+1} [\![P, i_{\phi_0} P']\!]^{\phi_0}. \tag{3.24}$$

On the other hand, we have that

$$[\![P, f]\!]^{\phi_0} = i_{(d^{\phi_0} f)} P, \tag{3.25}$$

for $f \in C^\infty(M, \mathbb{R})$. From (3.23), (3.24) and (3.25), we obtain that

$$[\![f, [\![P', P'']\!]^{\phi_0}]\!]^{\phi_0} + [\![[\![f, P']\!]^{\phi_0}, P'']\!]^{\phi_0} + (-1)^{k'} [\![P', [\![f, P'']\!]^{\phi_0}]\!]^{\phi_0} = 0. \tag{3.26}$$

This proves (3.21) for $k = 0$.

On the other hand, if $X \in \Gamma(A)$, using (3.23) and the properties of the Schouten bracket $[\![\,,\,]\!]$, it follows that

$$[\![X, [\![P', P'']\!]^{\phi_0}]\!]^{\phi_0} = [\![[\![X, P']\!]^{\phi_0}, P'']\!]^{\phi_0} + [\![P', [\![X, P'']\!]^{\phi_0}]\!]^{\phi_0}. \tag{3.27}$$

We must show that (3.21) holds, for $k \geq 1$. But, this is equivalent to prove that (3.21) holds for $P' \in \Gamma(\wedge^{k'} A)$, $P'' \in \Gamma(\wedge^{k''} A)$ and $P = \bar{P} \wedge Y$, with $\bar{P} \in \Gamma(\wedge^{k-1} A)$ and $Y \in \Gamma(A)$.

We will proceed by induction on k. From (3.27), we deduce that the result is true for $k = 1$. Now, assume that

$$(-1)^{(\bar{k}+1)k''} [\![[\![\bar{Q} \wedge Y, P']\!]^{\phi_0}, P'']\!]^{\phi_0} + (-1)^{k'k''} [\![[\![P'', \bar{Q} \wedge Y]\!]^{\phi_0}, P']\!]^{\phi_0} + \\ (-1)^{(\bar{k}+1)k'} [\![[\![P', P'']\!]^{\phi_0}, \bar{Q} \wedge Y]\!]^{\phi_0} = 0,$$

for $\bar{Q} \in \Gamma(\wedge^{\bar{k}} A)$, with $\bar{k} \leq k - 2$.

Then, we have that

$$(-1)^{\tilde{k}k''}[\![[\![\tilde{Q},P']\!]^{\phi_0},P'']\!]^{\phi_0}+(-1)^{k'k''}[\![[\![P'',\tilde{Q}]\!]^{\phi_0},P']\!]^{\phi_0}$$
$$+(-1)^{\tilde{k}k'}[\![[\![P',P'']\!]^{\phi_0},\tilde{Q}]\!]^{\phi_0}=0,$$

for $\tilde{Q}\in\Gamma(\wedge^{\tilde{k}}A)$, with $\tilde{k}\leq k-1$.

Using this fact, (3.24) and (3.27), we conclude that

$$(-1)^{kk''}[\![[\![\bar{P}\wedge Y,P']\!]^{\phi_0},P'']\!]^{\phi_0}+(-1)^{k'k''}[\![[\![P'',\bar{P}\wedge Y]\!]^{\phi_0},P']\!]^{\phi_0}$$
$$+(-1)^{kk'}[\![[\![P',P'']\!]^{\phi_0},\bar{P}\wedge Y]\!]^{\phi_0}=0.$$

Finally, if $[\![\, ,\,]\!]\tilde{}:\Gamma(\wedge^*A)\times\Gamma(\wedge^*A)\to\Gamma(\wedge^*A)$ is an operation which satisfies (3.16)-(3.20), then it is clear that $[\![\, ,\,]\!]\tilde{}=[\![\, ,\,]\!]^{\phi_0}$. QED

The operation $[\![\, ,\,]\!]^{\phi_0}$ is called the ϕ_0-*Schouten bracket* of $(A,([\![\, ,\,]\!],\rho),\phi_0)$.

Remark 3.5 The ϕ_0-Schouten bracket of the Jacobi algebroid $(A,([\![\, ,\,]\!],\rho),\phi_0)$ can be characterized as follows. The product manifold $\bar{A}=A\times T\mathbb{R}$ is a vector bundle over $M\times\mathbb{R}$ and one may define a Lie algebroid structure $([\![\, ,\,]\!]\bar{},\bar{\rho})$ on $\bar{A}$, where $[\![\, ,\,]\!]\bar{}$ is the obvious product Lie bracket and $\bar{\rho}=\rho\times Id:\bar{A}\to TM\times T\mathbb{R}$. The direct sum $\Gamma(\wedge^kA)\oplus\Gamma(\wedge^{k-1}A)$ is a subspace of $\Gamma(\wedge^k\bar{A})$ and we may consider the monomorphism of $C^\infty(M,\mathbb{R})$-modules $\bar{U}_{\phi_0}:\Gamma(\wedge^kA)\to\Gamma(\wedge^k\bar{A})$ given by $\bar{U}_{\phi_0}(P)=(e^{-(k-1)t}P,e^{-(k-1)t}i_{\phi_0}(P))$. Then, it is easy to prove that $\bar{U}_{\phi_0}([\![P,P']\!]^{\phi_0})=[\![\bar{U}_{\phi_0}(P),\bar{U}_{\phi_0}(P')]\!]\bar{}$, for $P\in\Gamma(\wedge^kA)$ and $P'\in\Gamma(\wedge^{k'}A)$ (see [33]). Here, $[\![\, ,\,]\!]\bar{}$ denotes the usual Schouten bracket of the Lie algebroid $(\bar{A},[\![\, ,\,]\!]\bar{},\bar{\rho})$

Now, if $(A,([\![\, ,\,]\!],\rho),\phi_0)$ is a Jacobi algebroid, $X\in\Gamma(A)$ and $P\in\Gamma(\wedge^kA)$, we can define the ϕ_0-*Lie derivative of* P *by* X as follows

$$\mathcal{L}^{\phi_0}_X(P)=[\![X,P]\!]^{\phi_0}. \tag{3.28}$$

Then, from Theorem 3.4, we deduce

Proposition 3.6 *Let* $(([\![\, ,\,]\!],\rho),\phi_0)$ *be a Jacobi algebroid structure on* $\tau:A\to M$. *If* $f\in C^\infty(M,\mathbb{R})$, $X\in\Gamma(A)$, $P\in\Gamma(\wedge^kA)$ *and* $P'\in\Gamma(\wedge^{k'}A)$, *we have*

$$\mathcal{L}^{\phi_0}_X(P\wedge P')=(\mathcal{L}^{\phi_0}_X(P))\wedge P'+P\wedge(\mathcal{L}^{\phi_0}_X(P'))-\phi_0(X)P\wedge P', \tag{3.29}$$

$$\mathcal{L}^{\phi_0}_{fX}(P) = f\mathcal{L}^{\phi_0}_X(P) - X \wedge i_{df}P. \tag{3.30}$$

Finally, using (3.4), (3.18), (3.28) and (3.29), we obtain that

$$\mathcal{L}^{\phi_0}_X(i_\omega P) = i_P\left(\mathcal{L}^{\phi_0}_X\omega\right) + i_\omega\left(\mathcal{L}^{\phi_0}_X(P)\right) + (k-1)\phi_0(X)i_\omega P,$$

for $\omega \in \Gamma(\wedge^k A^*)$, $P \in \Gamma(\wedge^k A)$ and $X \in \Gamma(A)$.

3.2 Jacobi structures and Lie bialgebroids

Let $(A, [\![\ ,\ ]\!], \rho)$ be a Lie algebroid over a manifold M such that its dual bundle $A^* \to M$ also admits a Lie algebroid structure $([\![\ ,\ ]\!]_*, \rho_*)$. Then the pair (A, A^*) is said to be a *Lie bialgebroid* if

$$d_*[\![X,Y]\!] = [\![X, d_*Y]\!] - [\![Y, d_*X]\!], \tag{3.31}$$

for $X, Y \in \Gamma(A)$, where d_* denotes the differential associated with the Lie algebroid structure $([\![\ ,\ ]\!]_*, \rho_*)$ on A^* (see [83]).

Examples 3.7 **1.-** Let (M, Π) be a Poisson manifold and $(T^*M, [\![\ ,\ ]\!]_\Pi, \#_\Pi)$ be the associated Lie algebroid (see Example 4 in Section 1.2.2). If on TM we consider the trivial Lie algebroid structure then, using (1.32) and the properties of the Schouten-Nijenhuis bracket, we deduce that the pair (TM, T^*M) is a Lie bialgebroid (see also [83]).

2.- It is obvious that if $\mathfrak{g}$ is a Lie algebra, (3.31) reduces to Drinfeld's cocycle condition for the pair $(\mathfrak{g}, \mathfrak{g}^*)$ to be a Lie bialgebra [27].

3.- If Π' is a 2-vector on a manifold M, we will denote by $[\![\ ,\ ]\!]_{\Pi'}$ the skew-symmetric bracket defined by

$$[\![\mu,\nu]\!]_{\Pi'} = (\mathcal{L}_0)_{\#_{\Pi'}(\mu)}\nu - (\mathcal{L}_0)_{\#_{\Pi'}(\nu)}\mu - d_0(\Pi'(\mu,\nu)),$$

for $\mu, \nu \in \Omega^1(M)$.

Now, suppose that Π is a Poisson structure on M and that $\mathcal{N} : TM \to TM$ is a Nijenhuis operator on TM. Assume also that

$$\mathcal{N} \circ \#_\Pi = \#_\Pi \circ \mathcal{N}^*, \tag{3.32}$$

$\mathcal{N}^* : T^*M \to T^*M$ being the adjoint operator of $\mathcal{N}$. Then, one may consider the 2-vector Π_1 on M characterized by the condition $\#_{\Pi_1} = \#_\Pi \circ \mathcal{N}^*$ and the concomitant $C(\Pi, \mathcal{N})$ of Π and $\mathcal{N}$, that is, $C(\Pi, \mathcal{N})$ is the tensor field of type (2,1) defined by

$$C(\Pi, \mathcal{N})(\mu, \nu) = [\![\mu, \nu]\!]_{\Pi_1} - [\![\mathcal{N}^*\mu, \nu]\!]_\Pi - [\![\mu, \mathcal{N}^*\nu]\!]_\Pi + \mathcal{N}^*[\![\mu, \nu]\!]_\Pi,$$

for $\mu, \nu \in \Omega^1(M)$. The pair $(\Pi, \mathcal{N})$ is said to be a *Poisson-Nijenhuis structure* on M if (3.32) holds and $C(\Pi, \mathcal{N})$ identically vanishes (see [64]). In [62], it was proved that $(\Pi, \mathcal{N})$ is a Poisson-Nijenhuis structure on M if and only if the pair (TM, T^*M) is a Lie bialgebroid, when TM (respectively, T^*M) is equipped with the deformed Lie algebroid structure $([\, ,\,]_\mathcal{N}, \mathcal{N})$ (respectively, the cotangent Lie algebroid structure $([\![\, ,\,]\!]_\Pi, \#_\Pi)$).

In [61], it was given an equivalent definition of a Lie bialgebroid in terms of derivations of graded Lie algebras. Let (A, A^*) be a pair of Lie algebroids in duality. If d_* is the differential of $(A^*, [\![\, ,\,]\!]_*, \rho_*)$ and $[\![\, ,\,]\!]'$ is the modified Schouten bracket of A (see Remark 1.7), one can show that (A, A^*) is a Lie bialgebroid if and only if d_* is a derivation with respect to $(\oplus_k \Gamma(\wedge^k A), [\![\, ,\,]\!]')$, that is,

$$d_*[\![P, Q]\!]' = [\![d_*P, Q]\!]' + (-1)^{p+1}[\![P, d_*Q]\!]'$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$.

Next, suppose that (M, Λ, E) is a Jacobi manifold. We consider the 1-jet Lie algebroid $(T^*M \times \mathbb{R}, [\![\, ,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ associated with the Jacobi structure (Λ, E) and the 1-cocycle $(-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$. As we know, the dual bundle $TM \times \mathbb{R}$ admits a Lie algebroid structure $([\ \, ,\ ], \pi)$ and the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle (see Examples 3 and 4 in Section 1.2.2).

For the above Jacobi algebroids, we deduce

Proposition 3.8 *i) If $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$, then*

$$d_*^{(-E,0)}[(X, f), (Y, g)] = [(X, f), d_*^{(-E,0)}(Y, g)]^{(0,1)} - [(Y, g), d_*^{(-E,0)}(X, f)]^{(0,1)}.$$

ii) If $\mathcal{L}_*$ *denotes the Lie derivative on the Lie algebroid* $(T^*M \times \mathbb{R}, [\![\, ,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$*, then*

$$(\mathcal{L}_*^{(-E,0)})_{(0,1)}(P,Q) + (\tilde{\mathcal{L}}_0^{(0,1)})_{(-E,0)}(P,Q) = 0,$$

for $(P,Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M) \cong \Gamma(\wedge^k(TM \times \mathbb{R}))$.

Proof: i) It follows from (3.15), (3.19) and (3.21).

ii) Using (1.24), (1.30), (3.3), (3.12) and (3.28), we have that

$$\begin{aligned}
&(\mathcal{L}_*^{(-E,0)})_{(0,1)}(P,Q) + (\tilde{\mathcal{L}}_0^{(0,1)})_{(-E,0)}(P,Q) \\
&\quad = d_*^{(-E,0)}(Q,0) + i_{(0,1)}\Big(-[\Lambda, P] + (k-1)E \wedge P + \Lambda \wedge Q, [\Lambda, Q] \\
&\qquad -(k-2)E \wedge Q + [E,P]\Big) - \Big([E,P],[E,Q]\Big) = 0.
\end{aligned}$$

QED

Remark 3.9 If (M, Λ, E) is a Jacobi manifold and on $TM \times \mathbb{R}$ (respectively, $T^*M \times \mathbb{R}$) we consider the Lie algebroid structure $([\ ,\ ], \pi)$ (respectively, $([\![\, ,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$) then, from Proposition 3.8, we deduce that the pair $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$ is not, in general, a Lie bialgebroid (see also [111]).

3.3 Jacobi bialgebroids

Let A be a vector bundle over M and A^* the dual bundle to A. Suppose that $(([\![\, ,\,]\!], \rho), \phi_0)$ (respectively, $(([\![\, ,\,]\!]_*, \rho_*), X_0)$) is a Jacobi algebroid structure on A (respectively, A^*). Then, we will use the following notation:

- d (resp. d_*) is the differential of $(A, [\![\, ,\,]\!], \rho)$ (resp. $(A^*, [\![\, ,\,]\!]_*, \rho_*)$).
- d^{ϕ_0} (resp. $d_*^{X_0}$) is the ϕ_0-differential (resp. X_0-differential) of A (resp. A^*).
- $\mathcal{L}$ (resp. $\mathcal{L}_*$) is the Lie derivative of A (resp. A^*).
- $\mathcal{L}^{\phi_0}$ (resp. $\mathcal{L}_*^{X_0}$) is the ϕ_0-Lie derivative (resp. X_0-Lie derivative).
- $[\![\, ,\,]\!]^{\phi_0}$ (resp. $[\![\, ,\,]\!]_*^{X_0}$) is the ϕ_0-Schouten bracket (resp. X_0-Schouten bracket) on $(A, ([\![\, ,\,]\!], \rho), \phi_0)$ (resp. $(A^*, ([\![\, ,\,]\!]_*, \rho_*), X_0)$).
- $\rho^{\phi_0} : \Gamma(A) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ (resp. $\rho_*^{X_0} : \Gamma(A^*) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$) is the representation given by (3.1).

• $(\rho,\phi_0):\Gamma(A)\to\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$ (resp. $(\rho_*,X_0):\Gamma(A^*)\to\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$) is the homomorphism of $C^\infty(M,\mathbb{R})$-modules given by (3.9) and $(\rho,\phi_0)^*:\Omega^1(M)\times C^\infty(M,\mathbb{R})\to\Gamma(A^*)$ (resp. $(\rho_*,X_0)^*:\Omega^1(M)\times C^\infty(M,\mathbb{R})\to\Gamma(A)$) is the adjoint operator of (ρ,ϕ_0) (resp. (ρ_*,X_0)).

Now, Proposition 3.8 suggests us to introduce the following definition.

Definition 3.10 *The pair $((A,\phi_0),(A^*,X_0))$ is said to be a* Jacobi bialgebroid *over M if*

$$d_*^{X_0}[\![X,Y]\!]=[\![X,d_*^{X_0}Y]\!]^{\phi_0}-[\![Y,d_*^{X_0}X]\!]^{\phi_0},\tag{3.33}$$

$$(\mathcal{L}_*^{X_0})_{\phi_0}P+\mathcal{L}_{X_0}^{\phi_0}P=0,\tag{3.34}$$

for all $X,Y\in\Gamma(A)$ and $P\in\Gamma(\wedge^kA)$.

Using (3.1), (3.4), (3.18), (3.28) and (3.29), we obtain that (3.34) holds if and only if

$$\phi_0(X_0)=0,\quad \rho(X_0)=-\rho_*(\phi_0),\tag{3.35}$$

$$(\mathcal{L}_*)_{\phi_0}X+[\![X_0,X]\!]=0,\ \text{for } X\in\Gamma(A).\tag{3.36}$$

Note that (3.35) and (3.36) follow applying (3.34) to $P=f\in C^\infty(M,\mathbb{R})=\Gamma(\wedge^0A)$ and $P=X\in\Gamma(A)$, respectively.

Next, we will see that the base space of a Jacobi bialgebroid carries an induced Jacobi structure.

First, we will prove some results.

Proposition 3.11 *Let $((A,\phi_0),(A^*,X_0))$ be a Jacobi bialgebroid. Then,*

$$(\mathcal{L}_*^{X_0})_{d^{\phi_0}f}X=[\![X,d_*^{X_0}f]\!],\tag{3.37}$$

for $X\in\Gamma(A)$ and $f\in C^\infty(M,\mathbb{R})$.

Proof: Using (3.2) and the derivation law on Lie algebroids, we obtain that

$$\begin{aligned}d_*^{X_0}\Big([\![X,fY]\!]\Big)&=(d_*^{X_0}f)\wedge[\![X,Y]\!]+fd_*^{X_0}[\![X,Y]\!]-fX_0\wedge[\![X,Y]\!]\\&\quad+d_*^{X_0}(\rho(X)(f))\wedge Y+\rho(X)(f)d_*^{X_0}Y-\rho(X)(f)X_0\wedge Y,\end{aligned}$$

for $X, Y \in \Gamma(A)$ and $f \in C^\infty(M,\mathbb{R})$.
On the other hand, from (3.2), (3.29), (3.30) and (3.33), we deduce that

$$\begin{aligned} d_*^{X_0}\Big([\![X, fY]\!]\Big) &= \mathcal{L}_X^{\phi_0}(d_*^{X_0}(fY)) - \mathcal{L}_{fY}^{\phi_0}(d_*^{X_0}(X)) \\ &= \Big(\mathcal{L}_X^{\phi_0}(d_*^{X_0}f)\Big) \wedge Y + (d_*^{X_0}f) \wedge \mathcal{L}_X^{\phi_0}Y \\ &\quad -\phi_0(X)(d_*^{X_0}f)\wedge Y + f\mathcal{L}_X^{\phi_0}(d_*^{X_0}Y) + \rho(X)(f)d_*^{X_0}Y \\ &\quad -f\Big(\mathcal{L}_X^{\phi_0}X_0 \wedge Y + X_0 \wedge \mathcal{L}_X^{\phi_0}Y - \phi_0(X)X_0 \wedge Y\Big) \\ &\quad -\rho(X)(f)X_0 \wedge Y - f\mathcal{L}_Y^{\phi_0}(d_*^{X_0}X) - i_{df}(d_*^{X_0}X) \wedge Y. \end{aligned}$$

Thus, using again (3.33), it follows that

$$\begin{aligned} d_*^{X_0}(\rho(X)(f)) \wedge Y &= \Big(\mathcal{L}_X^{\phi_0}d_*^{X_0}f - \phi_0(X)d_*^{X_0}f - f\mathcal{L}_X^{\phi_0}X_0 \\ &\quad +f\phi_0(X)X_0 - i_{df}(d_*^{X_0}X)\Big) \wedge Y, \end{aligned}$$

and so

$$\begin{aligned} d_*^{X_0}(\rho(X)(f)) - \mathcal{L}_X^{\phi_0}d_*^{X_0}f + \phi_0(X)d_*^{X_0}f + f\mathcal{L}_X^{\phi_0}X_0 \\ - f\phi_0(X)X_0 + i_{df}(d_*^{X_0}X) = 0, \end{aligned}$$

which, by (3.2), (3.3) and (3.36), implies (3.37). *QED*

Corollary 3.12 *Under the same hypothesis as in Proposition 3.11, we have*

$$d^{\phi_0}f \cdot d_*^{X_0}g + d^{\phi_0}g \cdot d_*^{X_0}f = 0, \text{ for all } f, g \in C^\infty(M,\mathbb{R}). \tag{3.38}$$

Proof: First of all, we claim that

$$[\![d_*^{X_0}g, d_*^{X_0}f]\!] = d_*^{X_0}\Big(d^{\phi_0}f \cdot d_*^{X_0}g\Big), \tag{3.39}$$

for $f, g \in C^\infty(M,\mathbb{R})$. In fact, if $\tilde{d}_0^{(0,1)}$ is the operator defined by (3.6) then, from (3.1), (3.3), (3.4), (3.9), (3.11) and Proposition 3.11, we get that

$$\begin{aligned} [\![d_*^{X_0}g, d_*^{X_0}f]\!] &= (\mathcal{L}_*^{X_0})_{d^{\phi_0}f}(d_*^{X_0}g) = d_*^{X_0}\Big((\mathcal{L}_*^{X_0})_{d^{\phi_0}f}(g)\Big) \\ &= d_*^{X_0}\Big(\rho_*^{X_0}(d^{\phi_0}f)(g)\Big) = d_*^{X_0}\Big(\tilde{d}_0^{(0,1)}g \cdot (\rho_*, X_0)(d^{\phi_0}f)\Big) \\ &= d_*^{X_0}\Big(d^{\phi_0}f \cdot (\rho_*, X_0)^*(\tilde{d}_0^{(0,1)}g)\Big) = d_*^{X_0}\Big(d^{\phi_0}f \cdot d_*^{X_0}g\Big). \end{aligned}$$

Moreover, using (3.6), (3.9) and (3.11), it follows that

$$\begin{aligned} d^{\phi_0} f \cdot d_*^{X_0} g &= \tilde{d}_0^{(0,1)} f \cdot \Big((\rho,\phi_0)\circ(\rho_*,X_0)^*\Big)(\tilde{d}_0^{(0,1)} g) \\ &= \pi^{(0,1)}\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(\tilde{d}_0^{(0,1)} g), f\Big), \end{aligned} \tag{3.40}$$

where $\pi^{(0,1)} : (\mathfrak{X}(M)\times C^\infty(M,\mathbb{R}))\times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ is the representation given by (3.5).

Now, it is clear that (3.38) is equivalent to the condition

$$d^{\phi_0} f \cdot d_*^{X_0} f = 0, \text{ for all } f \in C^\infty(M,\mathbb{R}). \tag{3.41}$$

In order to prove (3.41), we first show that

$$\begin{aligned} &\tilde{d}_0^{(0,1)} f \cdot \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(0,1)\Big) \\ &\qquad = -\tilde{d}_0^{(0,1)} 1 \cdot \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0 f,0)\Big), \end{aligned} \tag{3.42}$$

for $f \in C^\infty(M,\mathbb{R})$. We have that (see (3.6), (3.9) and (3.35))

$$\begin{aligned} \tilde{d}_0^{(0,1)} f \cdot \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(0,1)\Big) &= (d_0 f, f)\cdot(\rho(X_0),0) \\ &= \rho(X_0)(f) = -\rho_*(\phi_0)(f). \end{aligned}$$

On the other hand, from (3.9) and (3.35), we deduce that

$$\begin{aligned} -\tilde{d}_0^{(0,1)} 1 \cdot \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0 f,0)\Big) &= -\phi_0\cdot(\rho_*,X_0)^*(d_0 f,0) \\ &= -\rho_*(\phi_0)(f). \end{aligned}$$

Thus, we deduce (3.42). Therefore, using (3.6), (3.40) and (3.42), we obtain that

$$d^{\phi_0} f \cdot d_*^{X_0} f = \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0 f,0)\Big)\cdot(d_0 f,0). \tag{3.43}$$

Now, we will prove that

$$\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0 f,0)\Big)\cdot(d_0 f,0) = 0.$$

From (3.2), (3.39), (3.40) and (3.43) it follows that $d_*^{X_0}\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f^2,0)\cdot(d_0f^2,0)\Big)=0$. Then,

$$\begin{aligned} 0 &= \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f,0)\cdot(d_0f,0)\Big)(d_*^{X_0}f^2-f^2X_0)\\ &= \Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f,0)\cdot(d_0f,0)\Big)d_*f^2 \qquad (3.44)\\ &= 2f\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f,0)\cdot(d_0f,0)\Big)d_*f. \end{aligned}$$

On the other hand, in general, $d_*g=(\rho_*,X_0)^*(d_0g,0)$. Thus, using (3.44),

$$f\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f,0)\cdot(d_0f,0)\Big)^2=0,\quad \text{for all } f.$$

This implies that

$$\Big(((\rho,\phi_0)\circ(\rho_*,X_0)^*)(d_0f,0)\Big)\cdot(d_0f,0)=0,$$

as we wanted to prove. Therefore, we conclude that $d^{\phi_0}f\cdot d_*^{X_0}f=0$, for all $f\in C^\infty(M,\mathbb{R})$, that is, (3.38) holds. QED

Next, we will show that if $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid over M, then M carries an induced Jacobi structure.

Theorem 3.13 *Let $((A,\phi_0),(A^*,X_0))$ be a Jacobi bialgebroid. Then, the bracket of functions $\{\,,\,\}_0 : C^\infty(M,\mathbb{R})\times C^\infty(M,\mathbb{R})\to C^\infty(M,\mathbb{R})$ given by*

$$\{f,g\}_0 := d^{\phi_0}f\cdot d_*^{X_0}g,\ \textit{for } f,g\in C^\infty(M,\mathbb{R}),$$

defines a Jacobi structure on M.

Proof: First of all, from Corollary 3.12 we obtain that the bracket $\{\,,\,\}_0$ is skew-symmetric.

From (3.2) and since $d^{\phi_0}1=\phi_0$, we deduce that $\{\,,\,\}_0$ is a first-order differential operator on each of its arguments with respect to the usual multiplication of functions.

Now, let us prove the Jacobi identity. Using (3.39), we have that

$$d^{\phi_0}h\cdot[\![d_*^{X_0}g,d_*^{X_0}f]\!]=d^{\phi_0}h\cdot d_*^{X_0}(\{f,g\}_0).$$

Thus, from (3.10) and (3.11), we deduce that

$$\tilde{d}_0^{(0,1)}h \cdot [(\rho,\phi_0)((\rho_*,X_0)^*(\tilde{d}_0^{(0,1)}g)),(\rho,\phi_0)((\rho_*,X_0)^*(\tilde{d}_0^{(0,1)}f))] \\ = d^{\phi_0}h \cdot d_*^{X_0}(\{f,g\}_0),$$

or, equivalently,

$$\pi^{(0,1)}\Big([((\rho,\phi_0)\circ(\rho_*,X_0)^*)(\tilde{d}_0^{(0,1)}(g)),((\rho,\phi_0)\circ(\rho_*,X_0)^*)(\tilde{d}_0^{(0,1)}(f))],h\Big) \\ = d^{\phi_0}h \cdot d_*^{X_0}(\{f,g\}_0).$$

Consequently, since $\pi^{(0,1)}$ is a representation of the Lie algebra $(\mathfrak{X}(M)\times C^\infty(M,\mathbb{R}),[\ ,\ ])$ on the space $C^\infty(M,\mathbb{R})$, this implies that (see (3.40))

$$\{f,\{g,h\}_0\}_0+\{g,\{h,f\}_0\}_0+\{h,\{f,g\}_0\}_0=0. \qquad \boxed{QED}$$

From (3.2) and (3.35), we have that

$$\{f,g\}_0 = df\cdot d_*g - f\rho(X_0)(g)+g\rho(X_0)(f), \tag{3.45}$$

for $f,g\in C^\infty(M,\mathbb{R})$. Since the differential d is a derivation with respect to the usual multiplication of functions we have that the map $(f,g)\mapsto df\cdot d_*g$, for $f,g\in C^\infty(M,\mathbb{R})$, is also a derivation on each of its arguments. Thus, we can define the 2-vector $\Lambda_0\in\mathcal{V}^2(M)$ characterized by the relation

$$\Lambda_0(d_0f,d_0g)=df\cdot d_*g=-dg\cdot d_*f, \tag{3.46}$$

for $f,g\in C^\infty(M,\mathbb{R})$, and the vector field $E_0\in\mathfrak{X}(M)$ by

$$E_0=-\rho(X_0)=\rho_*(\phi_0). \tag{3.47}$$

From (3.45), we obtain that

$$\{f,g\}_0=\Lambda_0(d_0f,d_0g)+fE_0(g)-gE_0(f),$$

for $f,g\in C^\infty(M,\mathbb{R})$. Therefore, the pair (Λ_0,E_0) is the Jacobi structure induced by the Jacobi bracket $\{\ ,\ \}_0$.

Finally, we will present an interesting characterization of Jacobi bialgebroids which was proved by Grabowski and Marmo in [33].

For this purpose, we will use the following notation. If $(([\![\, ,\,]\!], \rho), \phi_0)$ is a Jacobi algebroid structure on A and $[\![\, ,\,]\!]^{\phi_0}$ is the ϕ_0-Schouten bracket, we will denote by $[\![\, ,\,]\!]'^{\phi_0}$ the bracket defined by

$$[\![P, Q]\!]'^{\phi_0} = (-1)^{p+1}[\![P, Q]\!]^{\phi_0},$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$.

Then, we have that

Theorem 3.14 *[33] Let $(A, ([\![\, ,\,]\!], \rho), \phi_0)$ be a Jacobi algebroid. Assume also that the dual bundle A^* admits a Jacobi algebroid structure $(([\![\, ,\,]\!]_*, \rho_*), X_0)$. Then, $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid if and only if $d_*^{X_0}$ is a derivation with respect to $(\oplus_k \Gamma(\wedge^k A), [\![\, ,\,]\!]'^{\phi_0})$, that is,*

$$d_*^{X_0}[\![P, Q]\!]'^{\phi_0} = [\![d_*^{X_0} P, Q]\!]'^{\phi_0} + (-1)^{p+1}[\![P, d_*^{X_0} Q]\!]'^{\phi_0} \tag{3.48}$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$.

Proof: Let us set

$$D(P, Q) = d_*^{X_0}[\![P, Q]\!]'^{\phi_0} - [\![d_*^{X_0} P, Q]\!]'^{\phi_0} - (-1)^{p+1}[\![P, d_*^{X_0} Q]\!]'^{\phi_0},$$

for $P \in \Gamma(\wedge^p A)$ and $Q \in \Gamma(\wedge^* A)$.

Using (3.1), Theorem 3.4 and the properties of the X_0-differential, we deduce that

$$\begin{aligned} &D(Q, P) = -(-1)^{(p-1)(q-1)} D(P, Q), \\ &D(P, Q \wedge R) = D(P, Q) \wedge R + (-1)^{pq} Q \wedge D(P, R) - D(P, 1) \wedge Q \wedge R, \end{aligned} \tag{3.49}$$

for $P \in \Gamma(\wedge^p A)$, $Q \in \Gamma(\wedge^q A)$ and $R \in \Gamma(\wedge^* A)$.

Now, suppose that $d_*^{X_0}$ is a derivation with respect to $(\oplus_k \Gamma(\wedge^k A), [\![\, ,\,]\!]'^{\phi_0})$ or, equivalently, that D identically vanishes. Then, it is clear that (3.33) holds. Moreover, from (3.25), we have that

$$[\![R, 1]\!]'^{\phi_0} = (-1)^{r+1} i_{\phi_0} R, \text{ for } R \in \Gamma(\wedge^r A).$$

Thus, using (3.3), (3.28) and (3.48), we deduce (3.34).
Conversely, assume that $((A,\phi_0),(A^*,X_0)$ is a Jacobi bialgebroid. Then, from (3.25) and Corollary 3.12, we obtain that

$$D(f,g)=0, \text{ for } f,g\in C^\infty(M,\mathbb{R}). \tag{3.50}$$

Furthermore, using (3.3), (3.25) and Proposition 3.11, it follows that

$$D(X,f)=0, \text{ for } X\in\Gamma(A) \text{ and } f\in C^\infty(M,\mathbb{R}). \tag{3.51}$$

On the other hand, from (3.33), we deduce that

$$D(X,Y)=0, \text{ for } X,Y\in\Gamma(A). \tag{3.52}$$

Finally, using (3.49)-(3.52), we conclude that D identically vanishes, which implies that $d_*^{X_0}$ is a derivation with respect to $(\oplus_k\Gamma(\wedge^k A),[\![\,,\,]\!]'^{\phi_0})$. QED

3.4 Examples of Jacobi bialgebroids

3.4.1 Lie bialgebroids

Suppose that the pair $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid where both 1-cocycles vanish, that is, $\phi_0=0$ and $X_0=0$.

In this particular case, (3.33) and (3.34) are equivalent to the condition (3.31). Thus, the pair $((A,0),(A^*,0))$ is a Jacobi bialgebroid if and only if the pair (A,A^*) is a Lie bialgebroid.

If (A,A^*) is a Lie bialgebroid then, by the previous result and Theorem 3.13, a Jacobi structure (Λ_0,E_0) can be defined on the base space M. Since $\phi_0=0$ and $X_0=0$, from (3.47) we deduce that $E_0=0$, that is, the Jacobi structure is Poisson, which implies a well known result (see [83]): given a Lie bialgebroid (A,A^*) over M, the base space M carries an induced Poisson structure.

3.4.2 The Jacobi bialgebroid associated with a Jacobi structure

Let M be an arbitrary manifold. As we know, the vector bundle $TM \times \mathbb{R} \to M$ admits a Lie algebroid structure $([\ ,\ ], \pi)$ and $\phi_0 = (0,1) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle of this Lie algebroid (see Example 3 in Section 1.2.2).

Now, suppose that (M, Λ, E) is a Jacobi manifold. We consider the 1-jet Lie algebroid $(T^*M \times \mathbb{R}, [\![\, ,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ associated with the Jacobi structure (Λ, E) and the 1-cocycle $(-E, 0) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$.

Using Proposition 3.8, we deduce that the pair $((TM \times \mathbb{R}, (0,1)),(T^*M \times \mathbb{R}, (-E, 0)))$ is a Jacobi bialgebroid.

Moreover, from (1.26), (1.30), (3.46) and (3.47), we obtain that the Jacobi structure (Λ_0, E_0) on the base space M is just (Λ, E).

3.4.3 Jacobi bialgebroids and strong Jacobi-Nijenhuis structures

Assume that Λ' and E' are a 2-vector and a vector field on a manifold M. Since $\Gamma(\wedge^2(TM\times\mathbb{R}))$ may be identified with the product $\mathcal{V}^2(M)\times\mathfrak{X}(M)$, the pair (Λ', E') may be considered as an element of $\Gamma(\wedge^2(TM \times \mathbb{R}))$ and thus we have defined the corresponding homomorphism $\#_{(\Lambda',E')} : \Gamma(T^*M \times \mathbb{R}) \cong \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \Gamma(TM \times \mathbb{R}) \cong \mathfrak{X}(M) \times C^\infty(M,\mathbb{R})$ which is given by

$$\#_{(\Lambda',E')}(\mu, f) = (\#_{\Lambda'}(\mu) + fE', -\mu(E')) = (\widetilde{\#}_{(\Lambda',E')}(\mu, f), -\mu(E')), \quad (3.53)$$

for $(\mu, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$.

On the other hand, we will denote by $[\![\, ,]\!]_{(\Lambda',E')} : (\Omega^1(M) \times C^\infty(M,\mathbb{R}))^2 \to \Omega^1(M) \times C^\infty(M,\mathbb{R})$ the skew-symmetric bracket defined by

$$\begin{aligned}&[\![(\mu, f), (\nu, g)]\!]_{(\Lambda',E')}\\ &\quad =((\mathcal{L}_0)_{\#_{\Lambda'}(\mu)}\nu-(\mathcal{L}_0)_{\#_{\Lambda'}(\nu)}\mu-d_0(\Lambda'(\mu,\nu))+f(\mathcal{L}_0)_{E'}\nu-g(\mathcal{L}_0)_{E'}\mu\\ &\qquad -i_{E'}(\mu\wedge\nu), \Lambda'(\nu,\mu)+\#_{\Lambda'}(\mu)(g)-\#_{\Lambda'}(\nu)(f)+fE'(g)-gE'(f)),\end{aligned}$$

for $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

Now, suppose that (Λ, E) is a Jacobi structure on M. Then, we may consider the Lie algebroid $(T^*M \times \mathbb{R}, [\![\, , \,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ and the pair $X_0{=}(-E, 0)$ is a 1-cocycle of this Lie algebroid. On the other hand, the pair $(0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle of the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ and the $(0, 1)$-differential $\tilde{d}_0^{(0,1)}$ of $TM \times \mathbb{R}$ is given by (3.6).

Thus, using (1.24), (1.28), (1.29), (3.6), (3.22) and (3.53), it follows that the X_0-Schouten bracket $[\![\, , \,]\!]^{X_0}_{(\Lambda,E)}$ satisfies the following relations:

$$[\![\tilde{d}_0^{(0,1)} f, g]\!]^{X_0}_{(\Lambda,E)} = \tilde{d}_0^{(0,1)} g \cdot (\#_{(\Lambda,E)}(\tilde{d}_0^{(0,1)} f)), \tag{3.54}$$

$$[\![(\mu, h), \tilde{d}_0^{(0,1)} f]\!]^{X_0}_{(\Lambda,E)} = [\![\tilde{d}_0^{(0,1)}(\mu, h), f]\!]^{X_0}_{(\Lambda,E)} + \tilde{d}_0^{(0,1)}([\![(\mu, h), f]\!]^{X_0}_{(\Lambda,E)}), \tag{3.55}$$

$$[\![\tilde{d}_0^{(0,1)}(\mu, h), \tilde{d}_0^{(0,1)} f]\!]^{X_0}_{(\Lambda,E)} = -\tilde{d}_0^{(0,1)}([\![\tilde{d}_0^{(0,1)}(\mu, h), f]\!]^{X_0}_{(\Lambda,E)}), \tag{3.56}$$

for $(\mu, h) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ and $f, g \in C^\infty(M, \mathbb{R})$.

Next, we consider a Nijenhuis operator $\mathcal{N}$ on the Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ], \pi)$. Therefore, we obtain the corresponding Lie algebroid $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi_{\mathcal{N}})$ (see (1.34)). Denote by $\mathcal{N}^* : T^*M \times \mathbb{R} \to T^*M \times \mathbb{R}$ the adjoint operator of $\mathcal{N}$ and by ϕ_0 the section of $T^*M \times \mathbb{R} \to M$ given by

$$\phi_0 = \mathcal{N}^*(0, 1).$$

Since $\mathcal{N}$ is a Nijenhuis operator on $(TM \times \mathbb{R}, [\ ,\ ], \pi)$, it follows that

$$\phi_0[(X, f), (Y, g)]_{\mathcal{N}} = (0, 1)[\mathcal{N}(X, f), \mathcal{N}(Y, g)],$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$, and consequently, using that $(0, 1)$ is a 1-cocycle of $(TM \times \mathbb{R}, [\ ,\ ], \pi)$, we obtain that ϕ_0 is a 1-cocycle of $(TM \times \mathbb{R}, [\ ,\ ]_{\mathcal{N}}, \pi_{\mathcal{N}})$. Moreover, if $d_{\mathcal{N}}^{\phi_0}$ is the ϕ_0-differential of the Jacobi algebroid $(([\ ,\ ]_{\mathcal{N}}, \pi_{\mathcal{N}}), \phi_0)$, then, from (1.24), (1.35), (1.36) and (3.6), we have that

$$d_{\mathcal{N}}^{\phi_0} f = \mathcal{N}^* \tilde{d}_0^{(0,1)} f, \tag{3.57}$$

$$d_{\mathcal{N}}^{\phi_0}(\mu, h) = i_{\mathcal{N}}(\tilde{d}_0^{(0,1)}(\mu, h)) - \tilde{d}_0^{(0,1)} \mathcal{N}^*(\mu, h), \tag{3.58}$$

$$d_{\mathcal{N}}^{\phi_0}\tilde{d}_0^{(0,1)}f = -\tilde{d}_0^{(0,1)}(\mathcal{N}^*\tilde{d}_0^{(0,1)}f), \tag{3.59}$$

for $(\mu, h) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ and $f \in C^\infty(M, \mathbb{R})$.

On the other hand, suppose that (Λ, E) and $\mathcal{N}$ satisfy that

$$\mathcal{N} \circ \#_{(\Lambda,E)} = \#_{(\Lambda,E)} \circ \mathcal{N}^*. \tag{3.60}$$

In this case, we can define the pair (Λ_0, E_0) formed by the 2-vector Λ_0 and the vector field E_0 characterized by

$$\#_{(\Lambda_0,E_0)} = \#_{(\Lambda,E)} \circ \mathcal{N}^*. \tag{3.61}$$

We say that the pair $((\Lambda, E), \mathcal{N})$ is a *strong Jacobi-Nijenhuis structure* if and only if (3.60) holds and the concomitant of (Λ, E) and $\mathcal{N}$, $C((\Lambda, E), \mathcal{N})$, identically vanishes, where $C((\Lambda, E), \mathcal{N})$ is given by

$$\begin{aligned} C((\Lambda, E), \mathcal{N})((\mu, f), (\nu, g)) & \\ = & [\![(\mu, f), (\nu, g)]\!]_{(\Lambda_0,E_0)} - [\![\mathcal{N}^*(\mu, f), (\nu, g)]\!]_{(\Lambda,E)} \\ & -[\![(\mu, f), \mathcal{N}^*(\nu, g)]\!]_{(\Lambda,E)} + \mathcal{N}^*[\![(\mu, f), (\nu, g)]\!]_{(\Lambda,E)}, \end{aligned} \tag{3.62}$$

for $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$.

Remark 3.15 In [88] is introduced the notion of a Jacobi-Nijenhuis structure imposing weaker conditions than we have adopted here. Moreover, in [97, 98] are established some local models of Jacobi-Nijenhuis manifolds and a reduction theorem is obtained. In a different direction, in [114] is given another relation between Jacobi structures and Nijenhuis operators. In addition, the author compares both approaches (see [114]).

Example 3.16 Suppose that (M, η) is a contact manifold with associated Jacobi structure (Λ, E) and that (Λ_0, E_0) is a Jacobi structure on M compatible with (Λ, E), that is, $(\Lambda+\Lambda_0, E+E_0)$ is a Jacobi structure (see [44, 95]). Since the homomorphism $\#_{(\Lambda,E)}$ given by (3.53) is, in this case, an isomorphism, let us consider the $C^\infty(M, \mathbb{R})$-linear map $\mathcal{N} = \#_{(\Lambda_0,E_0)} \circ (\#_{(\Lambda,E)})^{-1}$.

Then, using the results in [88], we deduce that $((\Lambda,E),\mathcal{N})$ is a strong Jacobi-Nijenhuis structure. An explicit example of the precedent construction is the following one.

Let M be the product manifold $T^*Q\times\mathbb{R}$, where Q is a smooth manifold of dimension m. Denote by η_Q the canonical contact 1-form on $T^*Q\times\mathbb{R}$ given by (2.10), by Π_0 the canonical cosymplectic structure on $T^*Q\times\mathbb{R}$ and by (Λ,E) the Jacobi structure on $T^*Q\times\mathbb{R}$ associated with η_Q. Then, from (2.11), we obtain that the Jacobi structure (Λ,E) and the Poisson structure Π_0 are compatible. Therefore, the pair $((\Lambda,E),\mathcal{N})$ is a strong Jacobi-Nijenhuis structure, where $\mathcal{N}:\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})\to\mathfrak{X}(M)\times C^\infty(M,\mathbb{R})$ is the map defined by

$$\mathcal{N}=\#_{(\Pi_0,0)}\circ(\#_{(\Lambda,E)})^{-1}.$$

Moreover, using (2.11), it follows that

$$\mathcal{N}=Id-\Big(\frac{\partial}{\partial t},0\Big)\otimes(d_0t,0)-(-\Delta_{T^*Q},1)\otimes(0,1),$$

Δ_{T^*Q} being the Liouville vector field of T^*Q.

Next, we relate strong Jacobi-Nijenhuis structures and Jacobi bialgebroids in the following result.

Theorem 3.17 *Let (Λ,E) be a Jacobi structure on a manifold M and $\mathcal{N}$ be a Nijenhuis operator on $TM\times\mathbb{R}$. Consider on $TM\times\mathbb{R}$ (respectively, $T^*M\times\mathbb{R}$) the Lie algebroid structure $([\ ,\ ]_{\mathcal{N}},\pi_{\mathcal{N}})$ (respectively, $([\![\ ,\ ]\!]_{(\Lambda,E)},\widetilde{\#}_{(\Lambda,E)})$). Then, $((\Lambda,E),\mathcal{N})$ is a strong Jacobi-Nijenhuis structure if and only if the pair $((TM\times\mathbb{R},\phi_0),(T^*M\times\mathbb{R},X_0))$ is a Jacobi bialgebroid, where ϕ_0 (respectively, X_0) is the 1-cocycle on $TM\times\mathbb{R}$ (respectively, $T^*M\times\mathbb{R}$) given by $\phi_0=\mathcal{N}^*(0,1)$ (respectively, $X_0=(-E,0)$).*

Proof: Let us set

$$\begin{aligned}D((\mu,\gamma),(\mu',\gamma')) \;=\;& d_{\mathcal{N}}^{\phi_0}[\![(\mu,\gamma),(\mu',\gamma')]\!]_{(\Lambda,E)}^{X_0}\\ &+[\![d_{\mathcal{N}}^{\phi_0}(\mu,\gamma),(\mu',\gamma')]\!]_{(\Lambda,E)}^{X_0}\\ &+(-1)^k[\![(\mu,\gamma),d_{\mathcal{N}}^{\phi_0}(\mu',\gamma')]\!]_{(\Lambda,E)}^{X_0}\end{aligned}\qquad(3.63)$$

for $(\mu, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ and $(\mu', \gamma') \in \Omega^*(M) \oplus \Omega^{*-1}(M)$.
Using (3.6) and (3.49), we deduce that $D = 0$ if and only if

$$D(f, g) = 0, \quad D(\tilde{d}_0^{(0,1)} f, g) = 0, \quad D(\tilde{d}_0^{(0,1)} f, \tilde{d}_0^{(0,1)} g) = 0 \tag{3.64}$$

for $f, g \in C^\infty(M, \mathbb{R})$. Note that if $(\mu, \gamma) \in \Omega^k(M) \oplus \Omega^{k-1}(M)$ then for every point x of M there exists an open subset U of M, $x \in U$, such that on U

$$(\mu, \gamma) = \sum_{i=1}^{r} f_1^i (\tilde{d}_0^{(0,1)} f_2^i) \wedge \ldots \wedge (\tilde{d}_0^{(0,1)} f_k^i),$$

with $f_j^i \in C^\infty(U, \mathbb{R})$, for all i and j.

Now suppose that f and g are real C^∞-differentiable functions on M. Then, using (1.29), (3.1), (3.17), (3.53), (3.57) and (3.63), we get that

$$D(f, g) = \tilde{d}_0^{(0,1)} g \cdot ((\#_{(\Lambda,E)} \circ \mathcal{N}^* - \mathcal{N} \circ \#_{(\Lambda,E)}) \tilde{d}_0^{(0,1)} f). \tag{3.65}$$

On the other hand, from (3.54), (3.55), (3.57), (3.59), (3.62) and (3.63), we have that

$$D(\tilde{d}_0^{(0,1)} f, g) = C((\Lambda, E), \mathcal{N})(\tilde{d}_0^{(0,1)} f, \tilde{d}_0^{(0,1)} g) + \tilde{d}_0^{(0,1)}(D(f, g)). \tag{3.66}$$

Finally, using (3.55), (3.56), (3.58), (3.59) and (3.63), we obtain that

$$D(\tilde{d}_0^{(0,1)} f, \tilde{d}_0^{(0,1)} g) = -\tilde{d}_0^{(0,1)}(C((\Lambda, E), \mathcal{N})(\tilde{d}_0^{(0,1)} f, \tilde{d}_0^{(0,1)} g)). \tag{3.67}$$

Therefore, from (3.65), (3.66), (3.67) and Theorem 3.14, we conclude the result. *QED*

Remark 3.18 Theorem 3.17 was proved, independently, by Nunes da Costa in [96] by using other techniques.

As a consequence of Theorem 3.17, we recover a result obtained, with weaker hypotheses, in [88, 98].

Corollary 3.19 *Let $((\Lambda, E), \mathcal{N})$ be a strong Jacobi-Nijenhuis structure on a manifold M. Then the 2-vector Λ_0 and the vector field E_0 characterized by (3.61) define a Jacobi structure on M.*

Proof: Since $((TM\times\mathbb{R},\phi_0),(T^*M\times\mathbb{R},X_0))$ is a Jacobi bialgebroid, we can define a Jacobi bracket $\{\,,\,\}_0$ on M given by

$$\{f,g\}_0 = d_{\mathcal{N}}^{\phi_0} f\cdot d_*^{X_0} g,$$

for $f,g\in C^\infty(M,\mathbb{R})$ (see Theorem 3.13). Using (3.57), we deduce that

$$\begin{aligned}\{f,g\}_0 &= \mathcal{N}^*(\tilde{d}_0^{(0,1)}f)\cdot d_*^{X_0}g = \tilde{d}_0^{(0,1)}g\cdot((\#_{(\Lambda,E)}\circ\mathcal{N}^*)(\tilde{d}_0^{(0,1)}f))\\ &= \Lambda_0(d_0f,d_0g)+fE_0(g)-gE_0(f).\end{aligned}$$

Therefore, we conclude our result. QED

3.4.4 Triangular Jacobi bialgebroids

Let $((\llbracket\,,\,\rrbracket,\rho),\phi_0)$ be a Jacobi algebroid structure on $\tau: A\to M$. Moreover, let $\mathcal{C}$ be a ϕ_0*-canonical section*, that is, $\mathcal{C}\in\Gamma(\wedge^2 A)$ and

$$\llbracket\mathcal{C},\mathcal{C}\rrbracket^{\phi_0}=0. \tag{3.68}$$

We shall discuss what happens on the dual bundle $A^*\to M$. Remark 3.2 *ii)* and Remark 3.3 *i)* suggest us to introduce the bracket $\llbracket\,,\,\rrbracket_{*\mathcal{C}}$ on $\Gamma(A^*)$ defined by

$$\begin{aligned}\llbracket\mu,\nu\rrbracket_{*\mathcal{C}} &= \mathcal{L}^{\phi_0}_{\#_{\mathcal{C}}(\mu)}\nu-\mathcal{L}^{\phi_0}_{\#_{\mathcal{C}}(\nu)}\mu-d^{\phi_0}(\mathcal{C}(\mu,\nu))\\ &= i_{\#_{\mathcal{C}}(\mu)}(d^{\phi_0}\nu)-i_{\#_{\mathcal{C}}(\nu)}(d^{\phi_0}\mu)+d^{\phi_0}(\mathcal{C}(\mu,\nu)),\end{aligned} \tag{3.69}$$

for $\mu,\nu\in\Gamma(A^*)$.

Theorem 3.20 *Let $(A,(\llbracket\,,\,\rrbracket,\rho),\phi_0)$ a Jacobi algebroid over M and $\mathcal{C}$ be a ϕ_0-canonical section of A. Then:*

i) The dual bundle $A^*\to M$ together with the bracket defined in (3.69) and the bundle map $\rho_{*\mathcal{C}}=\rho\circ\#_{\mathcal{C}}:A^*\to TM$ is a Lie algebroid.

ii) $X_0=-\#_{\mathcal{C}}(\phi_0)\in\Gamma(A)$ is a 1-cocycle of $(A^*,\llbracket\,,\,\rrbracket_{*\mathcal{C}},\rho_{*\mathcal{C}})$.

iii) The pair $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid.

Proof: First of all, define a linear map of degree 1 on the space $\Gamma(\wedge^* A) = \oplus_k \Gamma(\wedge^k A)$ by

$$d_* Q = -[\![Q, \mathcal{C}]\!]^{\phi_0} + (i_{\phi_0}\mathcal{C}) \wedge Q, \tag{3.70}$$

for $Q \in \Gamma(\wedge^* A)$.

From (3.19) and (3.20), we deduce that d_* is a derivation with respect to $(\oplus_k \Gamma(\wedge^k A), \wedge)$. Moreover, using (3.20), (3.21), (3.24) and (3.68), we obtain that $d_*^2 = 0$. Thus, the results in [64, 121] claim that the equations

$$\rho_*(\mu)(f) = \mu(d_* f),$$
$$[\![\mu, \nu]\!]_*(X) = \rho_*(\mu)(\nu(X)) - \rho_*(\nu)(\mu(X)) - d_* X(\mu, \nu),$$

for $\mu, \nu \in \Gamma(A^*)$, $X \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$, define the anchor map and the Lie bracket of a Lie algebroid structure on A^*.

A simple computation, using (3.4), (3.25), (3.69), (3.70) and the fact that

$$(\mathcal{L}_{\#_\mathcal{C}(\mu)}\nu - \mathcal{L}_{\#_\mathcal{C}(\nu)}\mu - d(\mathcal{C}(\mu, \nu)))(X)$$
$$= [\![\mathcal{C}, X]\!](\mu, \nu) + \rho(\#_\mathcal{C}(\mu))(\nu(X)) - \rho(\#_\mathcal{C}(\nu))(\mu(X)),$$

for $\mu, \nu \in \Gamma(A^*)$ and $X \in \Gamma(A)$, shows that $[\![\ ,\ ]\!]_* = [\![\ ,\ ]\!]_{*\mathcal{C}}$ and $\rho_* = \rho_{*\mathcal{C}}$. Thus, we have *i)*.

Moreover, from (3.21), (3.24), (3.70) and Theorem 3.14, we conclude that $X_0 = -i_{\phi_0}\mathcal{C}$ is a 1-cocycle of $(A^*, [\![\ ,\ ]\!]_{*\mathcal{C}}, \rho_{*\mathcal{C}})$ and the pair $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid. QED

Let $(A, ([\![\ ,\ ]\!], \rho), \phi_0)$ be a Jacobi algebroid. Suppose that $(([\![\ ,\ ]\!]_*, \rho_*), X_0)$ is a Jacobi algebroid structure on A^*. Moreover, assume that $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid. Then, the pair $((A, \phi_0), (A^*, X_0))$ is said to be a *triangular Jacobi bialgebroid* if there exists a ϕ_0-canonical section $\mathcal{C}$ of A such that

$$[\![\ ,\ ]\!]_* = [\![\ ,\ ]\!]_{*\mathcal{C}}, \quad \rho_* = \rho_{*_\mathcal{C}}, \quad X_0 = -\#_\mathcal{C}(\phi_0).$$

Let $((A, \phi_0), (A^*, X_0))$ be a triangular Jacobi bialgebroid over M and $\mathcal{C}$ the corresponding ϕ_0-canonical section of A. If (Λ_0, E_0) is the induced Jacobi structure on M, using (3.46), (3.47) and Theorem 3.20, it follows that

$$\Lambda_0(d_0 f, d_0 g) = \mathcal{C}(df, dg), \qquad E_0 = \rho(\#_{\mathcal{C}}(\phi_0)), \tag{3.71}$$

for $f, g \in C^\infty(M, \mathbb{R})$.

Examples 3.21 **1.-** Note that a triangular Jacobi bialgebroid $((A, \phi_0), (A^*, X_0))$ such that $\phi_0 = 0$ is just a *triangular Lie bialgebroid* (see [83]).

2.- If (M, Λ, E) is a Jacobi manifold then, using Remarks 3.2 and 3.3, we deduce that the pair $((TM \times \mathbb{R}, \phi_0 = (0,1)), (T^*M \times \mathbb{R},\ X_0 = (-E, 0)))$ is a triangular Jacobi bialgebroid. Note that, in this case, the ϕ_0-canonical section of $TM \times \mathbb{R} \to M$ is just the Jacobi structure (Λ, E).

3.- Let (M, Λ, E) be a Jacobi manifold. Suppose that there exists a closed 1-form θ such that $\#_\Lambda(\theta) = E$. Then, θ is a 1-cocycle of the trivial Lie algebroid $(TM, [\,,\,], Id)$ and we may consider the θ-Schouten bracket $[\,,\,]^\theta$. Moreover, from (1.1) and Theorem 3.4, we deduce that

$$[\Lambda, \Lambda]^\theta = [\Lambda, \Lambda] - 2\,(i_\theta \Lambda) \wedge \Lambda = 0.$$

Thus, Λ is a θ-canonical section of A. Therefore, using Theorem 3.20, we obtain that the pair $((TM, \theta), (T^*M, -E))$ is a triangular Jacobi bialgebroid, where the Lie algebroid structure $([\![\,,\,]\!]_{(\Lambda,E,\theta)}, \widetilde{\#}_{(\Lambda,E,\theta)})$ on T^*M is given by

$$\begin{aligned} [\![\mu, \nu]\!]_{(\Lambda,E,\theta)} &= (\mathcal{L}_0)_{\#_\Lambda(\mu)}\nu - (\mathcal{L}_0)_{\#_\Lambda(\nu)}\mu - d_0(\Lambda(\mu,\nu)) \\ &\quad - i_E(\mu \wedge \nu) - \Lambda(\mu,\nu)\theta, \\ \widetilde{\#}_{(\Lambda,E,\theta)}(\mu) &= \#_\Lambda(\mu), \end{aligned} \tag{3.72}$$

for $\mu, \nu \in \Omega^1(M)$.

Since $((TM, \theta),\ (T^*M, -E))$ is a Jacobi bialgebroid over M, M carries an induced Jacobi structure. In fact, using (3.71), we deduce that this Jacobi structure is just (Λ, E).

Remark 3.22 *i)* If (M, Ω) is a l.c.s. manifold with Lee 1-form ω and (Λ, E) is the associated Jacobi structure on M, we have that $\#_\Lambda(\omega) = -E$ (see (1.12), (1.13) and Remark 1.1).

ii) Let (Λ, E) be a Jacobi structure on M and θ be a closed 1-form on M such that $\#_\Lambda(\theta) = E$. It is clear that

$$\theta(E) = 0, \qquad (\mathcal{L}_0)_E\theta = 0. \tag{3.73}$$

Now, denote by $([\![\,,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ (respectively, $([\![\,,\,]\!]_{(\Lambda,E,\theta)}, \widetilde{\#}_{(\Lambda,E,\theta)})$) the Lie algebroid structure on $T^*M \times \mathbb{R}$ (respectively, T^*M) given by (1.29) (respectively, (3.72)) and by $\Psi : T^*M \times \mathbb{R} \to T^*M$ the epimorphism of vector bundles (over the identity $Id : M \to M$) defined by

$$\Psi(\mu_x, \lambda) = \mu_x + \lambda\,\theta_x, \text{ for } (\mu_x, \lambda) \in T_x^*M \times \mathbb{R}.$$

Then, using (1.29), (3.72), (3.73) and the fact that θ is a closed 1-form, we deduce that the pair (Ψ, Id) is an epimorphism between the Lie algebroids $(T^*M \times \mathbb{R}, [\![\,,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ and $(T^*M, [\![\,,\,]\!]_{(\Lambda,E,\theta)}, \widetilde{\#}_{(\Lambda,E,\theta)})$.

3.4.5 The Jacobi bialgebroid associated with an exact Poisson structure

Let M be an arbitrary manifold. We have seen that the triple $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ is a Lie algebroid over M (see Example 3 in Section 1.2.2). Evidently, we have that $\phi_0 = (0, 0) \in \Omega^1(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(T^*M \times \mathbb{R})$ is a 1-cocycle for this Lie algebroid.

In addition, suppose that there exists an exact Poisson structure on M, that is, there exists a 2-vector Π and a vector field Z on M such that

$$[\Pi, \Pi] = 0, \qquad [Z, \Pi] = -\Pi.$$

Then, the vector bundle $T^*M \times \mathbb{R} \to M$ admits a Lie algebroid structure $([\![\,,\,]\!]_{(\Pi,Z)}, \bar{\#}_{(\Pi,Z)})$ and $X_0{=}(0, 1) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$ is a 1-cocycle of this Lie algebroid structure (see Example 7 in Section 1.2.2).

Moreover, from (1.24), (1.38) and (3.2), we deduce that the X_0-differential is

$$d_*^{X_0}(P, Q) = (-[\Pi, P], [\Pi, Q] - [Z, P] - (k - 1)P), \tag{3.74}$$

for $(P, Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$.

Next, we will prove that the pair $((TM \times \mathbb{R}, (0,0)), (T^*M \times \mathbb{R}, (0,1)))$ is a Jacobi bialgebroid over M.

First of all, from (1.27), it follows that

$$\begin{aligned}(\mathcal{L}_*^{X_0})_{\phi_0}(P,Q) + \mathcal{L}_{X_0}^{\phi_0}(P,Q) &= [(0,1),(P,Q)] \\ &= ([0,P], -[0,Q] - [1,P]) = (0,0),\end{aligned}$$

for $(P,Q) \in \mathcal{V}^k(M) \oplus \mathcal{V}^{k-1}(M)$, that is, (3.34) holds.

On the other hand, an easy computation, using (1.27), (3.74) and the properties of the Schouten bracket, shows that

$$d_*^{X_0}[\![(X,f),(Y,g)]\!] = [\![(X,f), d_*^{X_0}(Y,g)]\!]^{\phi_0} - [\![(Y,g), d_*^{X_0}(X,f)]\!]^{\phi_0},$$

for $(X,f),(Y,g) \in \mathfrak{X}(M) \times C^\infty(M,\mathbb{R}) \cong \Gamma(TM \times \mathbb{R})$.

Finally, since $((TM \times \mathbb{R}, (0,0)), (T^*M \times \mathbb{R}, (0,1)))$ is a Jacobi bialgebroid over M, we know, by Theorem 3.13, that there exists a Jacobi structure on the base space M. In fact, if $\{\ ,\ \}_0$ is the Jacobi bracket on M and $f, g \in C^\infty(M,\mathbb{R})$ then (see (1.26) and (3.74))

$$\begin{aligned}\{f,g\}_0 &= \tilde{d}_0 f \cdot d_*^{X_0} g = \langle (d_0 f, 0), (-[g,\Pi], g - Z(g)) \rangle \\ &= \Pi(d_0 f, d_0 g),\end{aligned}$$

that is, the Jacobi structure is just the original exact Poisson structure Π.

3.5 Lie bialgebroids associated with Jacobi bialgebroids

In this Section, we will show that it is possible to construct a Lie bialgebroid from a Jacobi bialgebroid and, as a consequence, we deduce a duality theorem.

3.5.1 Time-dependent sections of a Lie algebroid

Let $(A, (\llbracket\,,\,\rrbracket, \rho), \phi_0)$ be a Jacobi algebroid over M and $\pi_1 : M \times \mathbb{R} \to M$ be the canonical projection over the first factor. We consider the map $* : \Gamma(A) \to \mathfrak{X}(M \times \mathbb{R})$ given by

$$X^* = \rho(X) + \phi_0(X)\frac{\partial}{\partial t}.$$

It is easy to prove that $*$ is an action of A on π_1 in the sense of Section 1.2.2 (see Example 6 in Section 1.2.2). Thus, if $\pi_1^* A$ is the pull-back of A over π_1 then the vector bundle $\pi_1^* A \to M \times \mathbb{R}$ admits a Lie algebroid structure $(\llbracket\,,\,\rrbracket^{-\phi_0}, \bar{\rho}^{\phi_0})$. For the sake of simplicity, when the 1-cocycle ϕ_0 is zero, we will denote by $(\llbracket\,,\,\rrbracket, \rho)$ the resultant Lie algebroid structure on $\pi_1^* A \to M \times \mathbb{R}$. On the other hand, it is clear that the vector bundles $\pi_1^* A \to M \times \mathbb{R}$ and $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ are isomorphic and that the space of sections $\Gamma(\bar{A})$ of $\bar{A} \to M \times \mathbb{R}$ can be identified with the set of time-dependent sections of $A \to M$. Under this identification, we have that $\llbracket\bar{X}, \bar{Y}\rrbracket(x,t) = \llbracket\bar{X}_t, \bar{Y}_t\rrbracket(x)$ and that $\rho(\bar{X})(x,t) = \rho(\bar{X}_t)(x)$, for $\bar{X}, \bar{Y} \in \Gamma(\bar{A})$ and $(x,t) \in M \times \mathbb{R}$. In addition,

$$\begin{aligned} \llbracket\bar{X}, \bar{Y}\rrbracket^{-\phi_0} &= \llbracket\bar{X}, \bar{Y}\rrbracket + \phi_0(\bar{X})\frac{\partial \bar{Y}}{\partial t} - \phi_0(\bar{Y})\frac{\partial \bar{X}}{\partial t}, \\ \bar{\rho}^{\phi_0}(\bar{X}) &= \rho(\bar{X}) + \phi_0(\bar{X})\frac{\partial}{\partial t}. \end{aligned} \tag{3.75}$$

Remark 3.23 *i)* Denote by d the differentials of the Lie algebroids $(A, \llbracket\,,\,\rrbracket, \rho)$ and $(\bar{A}, \llbracket\,,\,\rrbracket, \rho)$. Then, if $\bar{\mu} \in \Gamma(\wedge^k \bar{A}^*)$ and $(x,t) \in M \times \mathbb{R}$, $d\bar{\mu} \in \Gamma(\wedge^{k+1}\bar{A}^*)$ and

$$(d\bar{\mu})(x,t) = (d\bar{\mu}_t)(x).$$

We also denote by $\llbracket\,,\,\rrbracket$ the Schouten bracket of the Lie algebroid $(\bar{A}, \llbracket\,,\,\rrbracket, \rho)$.

ii) For any $\bar{P} \in \Gamma(\wedge^k \bar{A})$ or $\bar{\omega} \in \Gamma(\wedge^k \bar{A}^*)$, one can define its derivative with respect to the time

$$\frac{\partial \bar{P}}{\partial t} \in \Gamma(\wedge^k \bar{A}), \qquad \frac{\partial \bar{\mu}}{\partial t} \in \Gamma(\wedge^k \tilde{A}^*).$$

Thus, we have two $\mathbb{R}$-linear operators of degree zero

$$\frac{\partial}{\partial t} : \Gamma(\wedge^k \bar{A}) \to \Gamma(\wedge^k \bar{A}), \qquad \frac{\partial}{\partial t} : \Gamma(\wedge^k \bar{A}^*) \to \Gamma(\wedge^k \bar{A}^*),$$

which have the following properties

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}(\bar{P} \wedge \bar{Q}) = \dfrac{\partial \bar{P}}{\partial t} \wedge \bar{Q} + \bar{P} \wedge \dfrac{\partial \bar{Q}}{\partial t}, \\ \dfrac{\partial}{\partial t}(\bar{\mu} \wedge \bar{\nu}) = \dfrac{\partial \bar{\mu}}{\partial t} \wedge \bar{\nu} + \bar{\mu} \wedge \dfrac{\partial \bar{\nu}}{\partial t}, \end{array}\right\} \tag{3.76}$$

$$\frac{\partial}{\partial t}[\![\bar{P}, \bar{Q}]\!] = [\![\frac{\partial \bar{P}}{\partial t}, \bar{Q}]\!] + [\![\bar{P}, \frac{\partial \bar{Q}}{\partial t}]\!], \tag{3.77}$$

$$d\left(\frac{\partial \bar{\mu}}{\partial t}\right) = \frac{\partial}{\partial t}(d\bar{\mu}), \tag{3.78}$$

for $\bar{P} \in \Gamma(\wedge^k \bar{A})$, $\bar{Q} \in \Gamma(\wedge^r \bar{A})$, $\bar{\mu} \in \Gamma(\wedge^k \bar{A}^*)$ and $\bar{\nu} \in \Gamma(\wedge^r \bar{A}^*)$.

On the other hand, in Section 2.3 (see Example 6) we proved that the vector bundle $\bar{A} \to M \times \mathbb{R}$ admits a Lie algebroid structure $([\![\,,\,]\!]^{\hat{\ }\phi_0}, \hat{\rho}^{\phi_0})$ given by (2.12).

Now, the bundle map $\Theta : \bar{A} \to \bar{A}, (v,t) \mapsto (e^t v, t)$, is an isomorphism of vector bundles and

$$\hat{\rho}^{\phi_0} \circ \Theta = \bar{\rho}^{\phi_0}, \qquad \Theta[\![\bar{X}, \bar{Y}]\!]^{-\phi_0} = [\![\Theta\bar{X}, \Theta\bar{Y}]\!]^{\hat{\ }\phi_0}.$$

Thus,

Proposition 3.24 *Let $(A, ([\![\,,\,]\!], \rho), \phi_0)$ be a Jacobi algebroid. Then:*

i) The triples $(\bar{A}, [\![\,,\,]\!]^{\hat{\ }\phi_0}, \hat{\rho}^{\phi_0})$ and $(\bar{A}, [\![\,,\,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ are Lie algebroids over $M \times \mathbb{R}$.

ii) The map $\Theta : \bar{A} \to \bar{A}$ defines an isomorphism between the Lie algebroids $(\bar{A}, [\![\,,\,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ and $(\bar{A}, [\![\,,\,]\!]^{\hat{\ }\phi_0}, \hat{\rho}^{\phi_0})$.

Now, let $A \to M$ be a vector bundle over a manifold M and suppose that $[\![\, ,]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is a bracket on the space $\Gamma(A)$, that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism of $C^\infty(M, \mathbb{R})$-modules and that ϕ_0 is a section of the dual bundle A^*.

We can define the bracket $[\![\, ,]\!]^{\bar{}\phi_0} : \Gamma(\bar{A}) \times \Gamma(\bar{A}) \to \Gamma(\bar{A})$ on the space $\Gamma(\bar{A})$ and the homomorphism of $C^\infty(M \times \mathbb{R}, \mathbb{R})$-modules $\bar{\rho}^{\phi_0} : \Gamma(\bar{A}) \to \mathfrak{X}(M \times \mathbb{R})$ given by (3.75).

Proposition 3.25 *If the triple $(\bar{A}, [\![\, ,]\!]^{\bar{}\phi_0}, \bar{\rho}^{\phi_0})$ is a Lie algebroid on $M \times \mathbb{R}$ then the triple $(A, ([\![\, ,]\!], \rho), \phi_0)$ is a Jacobi algebroid on M.*

Proof: From (3.75), it follows that $[\![X, Y]\!]^{\bar{}\phi_0} = [\![X, Y]\!]$, for $X, Y \in \Gamma(A)$. Thus, we have that the bracket $[\![\, ,]\!]$ defines a Lie algebra structure on $\Gamma(A)$. On the other hand, if $f \in C^\infty(M, \mathbb{R})$ then, using (3.75) and the fact that $[\![X, fY]\!]^{\bar{}\phi_0} = f[\![X, Y]\!]^{\bar{}\phi_0} + (\bar{\rho}^{\phi_0}(X)(f))Y$, we obtain that

$$[\![X, fY]\!] = f[\![X, Y]\!] + (\rho(X)(f))Y.$$

Finally, since $\bar{\rho}^{\phi_0}[\![X, Y]\!]^{\bar{}\phi_0} = [\bar{\rho}^{\phi_0}(X), \bar{\rho}^{\phi_0}(Y)]$, we deduce that ϕ_0 is a 1-cocycle.

QED

From Propositions 3.24 and 3.25, we conclude

Proposition 3.26 *Let $A \to M$ be a vector bundle over a manifold M. Suppose that $[\![\, ,]\!] : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ is a bracket on the space $\Gamma(A)$, that $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a homomorphism of $C^\infty(M, \mathbb{R})$-modules and that ϕ_0 is a section of the dual bundle A^*. If $[\![\, ,]\!]^{\hat{}\phi_0} : \Gamma(\bar{A}) \times \Gamma(\bar{A}) \to \Gamma(\bar{A})$ and $\hat{\rho}^{\phi_0} : \Gamma(\bar{A}) \to \mathfrak{X}(M \times \mathbb{R})$ (respectively, $[\![\, ,]\!]^{\bar{}\phi_0} : \Gamma(\bar{A}) \times \Gamma(\bar{A}) \to \Gamma(\bar{A})$ and $\bar{\rho}^{\phi_0} : \Gamma(\bar{A}) \to \mathfrak{X}(M \times \mathbb{R})$) are the bracket on $\Gamma(\bar{A})$ and the homomorphism of $C^\infty(M \times \mathbb{R}, \mathbb{R})$-modules given by (2.12) (respectively, (3.75)) then the following conditions are equivalent:*

i) The triple $(A, ([\![\, ,]\!], \rho), \phi_0)$ is a Jacobi algebroid.

ii) The triple $(\bar{A}, [\![\, ,]\!]^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$ is a Lie algebroid.

iii) The triple $(\bar{A}, [\![\, , \,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ *is a Lie algebroid.*

Remark 3.27 Let $(A, ([\![\, , \,]\!], \rho), \phi_0)$ be a Jacobi algebroid over M. If $\bar{d}^{\phi_0}$ (respectively, $\hat{d}^{\phi_0}$) is the differential of the Lie algebroid $(\bar{A}, [\![\, , \,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ (respectively, $(\bar{A}, [\![,]\!]^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$), and $[\![\, , \,]\!]^{-\phi_0}$ is the Schouten bracket of the Lie algebroid $(\bar{A}, [\![\, , \,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$, then we have that

$$\bar{d}^{\phi_0}\bar{f} = d\bar{f} + \frac{\partial \bar{f}}{\partial t}\phi_0, \tag{3.79}$$

$$\bar{d}^{\phi_0}\bar{\phi} = d\bar{\phi} + \phi_0 \wedge \frac{\partial \bar{\phi}}{\partial t}, \tag{3.80}$$

$$[\![\bar{X}, \bar{P}]\!]^{-\phi_0} = [\![\bar{X}, \bar{P}]\!]^{\phi_0} + \phi_0(\bar{X})\left(\bar{P} + \frac{\partial \bar{P}}{\partial t}\right) - \frac{\partial \bar{X}}{\partial t} \wedge (i_{\phi_0}\bar{P}), \tag{3.81}$$

$$\hat{d}^{\phi_0}\bar{f} = e^{-t}\Big(d\bar{f} + \frac{\partial \bar{f}}{\partial t}\phi_0\Big), \tag{3.82}$$

$$\hat{d}^{\phi_0}\bar{\phi} = e^{-t}\left(d^{\phi_0}\bar{\phi} + \phi_0 \wedge \frac{\partial \bar{\phi}}{\partial t}\right), \tag{3.83}$$

for $\bar{f} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$, $\bar{\phi} \in \Gamma(\bar{A}^*)$, $\bar{X} \in \Gamma(\bar{A})$ and $\bar{P} \in \Gamma(\wedge^2 \bar{A})$.

3.5.2 Lie bialgebroids and Jacobi bialgebroids

First of all, we will prove a general result which will be useful in the sequel. Suppose that $(A_i, [\![\, , \,]\!]_i, \rho_i)$, $i = 1, 2$, are two Lie algebroids over M such that the dual bundles A_1^* and A_2^* are Lie algebroids with Lie algebroid structures $([\![\, , \,]\!]_{1*}, \rho_{1*})$ and $([\![\, , \,]\!]_{2*}, \rho_{2*})$, respectively.

Proposition 3.28 *Let* $\Phi : A_1 \to A_2$ *be a Lie algebroid isomorphism over the identity* $Id : M \to M$ *such that its adjoint homomorphism* $\Phi^* : A_2^* \to A_1^*$ *is also a Lie algebroid isomorphism. Then, if* (A_1, A_1^*) *is a Lie bialgebroid, so is* (A_2, A_2^*).

Proof: Denote also by $\Phi : \wedge^k A_1 \to \wedge^k A_2$ the isomorphism between the vector bundles $\wedge^k A_1 \to M$ and $\wedge^k A_2 \to M$ induced by $\Phi : A_1 \to A_2$. If

$\Phi : \Gamma(\wedge^k A_1) \to \Gamma(\wedge^k A_2)$ is the corresponding isomorphism of $C^\infty(M,\mathbb{R})$-modules, we have that

$$\Phi(P)(\mu_1,\dots,\mu_k) = P(\Phi^*\mu_1,\dots,\Phi^*\mu_k),$$

$$\Phi(X_1 \wedge \dots \wedge X_k) = \Phi(X_1) \wedge \dots \wedge \Phi(X_k),$$

for $P \in \Gamma(\wedge^k A_1), \mu_1,\dots,\mu_k \in \Gamma(A_2^*)$ and $X_1,\dots,X_k \in \Gamma(A_1)$. Thus, using that Φ and Φ^* are Lie algebroid isomorphisms, it follows that

$$d_{2*}(\Phi(X_1)) = \Phi(d_{1*}X_1), \qquad \Phi[\![X_1, P_1]\!]_1 = [\![\Phi(X_1), \Phi(P_1)]\!]_2, \tag{3.84}$$

for $X_1 \in \Gamma(A_1)$ and $P_1 \in \Gamma(\wedge^k A_1)$, where d_{1*} (resp. d_{2*}) is the differential of $(A_1^*, [\![\ ,\ ]\!]_{1*}, \rho_{1*})$ (resp. $(A_2^*, [\![\ ,\ ]\!]_{2*}, \rho_{2*})$).

Now, if $X_2, Y_2 \in \Gamma(A_2)$ then there exist $X_1, Y_1 \in \Gamma(A_1)$ such that $Y_i = \Phi(X_i)$, for $i = 1,2$. Therefore, from (3.84) and since (A_1, A_1^*) is a Lie bialgebroid, we obtain that

$$\begin{aligned} d_{2*}[\![X_2, Y_2]\!]_2 &= d_{2*}(\Phi[\![X_1, Y_1]\!]_1) = \Phi([\![X_1, d_{1*}Y_1]\!]_1 - [\![Y_1, d_{1*}X_1]\!]_1) \\ &= [\![X_2, d_{2*}Y_2]\!]_2 - [\![Y_2, d_{2*}X_2]\!]_2. \end{aligned}$$

Consequently, (A_2, A_2^*) is a Lie bialgebroid. *QED*

Next, assume that (M, Λ_0, E_0) is a Jacobi manifold. Consider on $A = TM \times \mathbb{R}$ and on $A^* = T^*M \times \mathbb{R}$ the Lie algebroid structures $([\ ,\ ], \pi)$ and $([\![\ ,\ ]\!]_{(\Lambda_0,E_0)}, \widetilde{\#}_{(\Lambda_0,E_0)})$, respectively. Then, the pair $((A, \phi_0 = (0,1)), (A^*, X_0 = (-E_0, 0)))$ is a Jacobi bialgebroid.

On the other hand, the map $\Phi : \bar{A} = A \times \mathbb{R} \to T(M \times \mathbb{R})$ given by

$$\Phi((v_{x_0}, \lambda_0), t_0) = v_{x_0} + \lambda_0 \frac{\partial}{\partial t}_{|t_0},$$

for $x_0 \in M$, $v_{x_0} \in T_{x_0}M$ and $\lambda_0, t_0 \in \mathbb{R}$, induces an isomorphism between the vector bundles $A \times \mathbb{R} \to M \times \mathbb{R}$ and $T(M \times \mathbb{R}) \to M \times \mathbb{R}$. Moreover, using (3.75), we deduce that Φ defines an isomorphism between the Lie algebroids

$(\bar{A},[\ ,\ ]^{-\phi_0},\bar{\pi}^{\phi_0})$ and $(T(M\times\mathbb{R}),[\,,\,],Id)$. Note that if $(\bar{X},\bar{f})$ is a time-dependent section of the vector bundle $TM\times\mathbb{R}\to M$ then $\dfrac{\partial(\bar{X},\bar{f})}{\partial t}$ is the time-dependent section given by $([\dfrac{\partial}{\partial t},\bar{X}],\dfrac{\partial\bar{f}}{\partial t})$.

Now, if $\bar{\mu}$ is a time-dependent 1-form on M, $\bar{X}$ is a time-dependent vector field and $(x_0,t_0)\in M\times\mathbb{R}$ then, using the isomorphism $T^*_{(x_0,t_0)}(M\times\mathbb{R})\cong T^*_{x_0}M\oplus T^*_{t_0}\mathbb{R}$, it follows that

$$((\bar{\mathcal{L}}_0)_{\bar{X}}\bar{\mu})_{(x_0,t_0)} = ((\mathcal{L}_0)_{\bar{X}_{t_0}}\bar{\mu}_{t_0})_{(x_0)} + \bar{\mu}_{(x_0,t_0)}\Big(\frac{\partial\bar{X}}{\partial t}_{|(x_0,t_0)}\Big)d_0t_{|t_0}, \tag{3.85}$$

where $\bar{\mathcal{L}}_0$ is the Lie derivative on $M\times\mathbb{R}$.

Moreover, $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\bar{\mu}$ is a time-dependent 1-form on M and if $\bar{f}\in C^\infty(M\times\mathbb{R},\mathbb{R})$, then $(\bar{\mu},\bar{f})$ is a time-dependent section of the vector bundle $T^*M\times\mathbb{R}\to M$ and

$$\frac{\partial(\bar{\mu},\bar{f})}{\partial t} = ((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\bar{\mu},\frac{\partial\bar{f}}{\partial t}). \tag{3.86}$$

A long computation, using (1.29), (1.31), (3.85) and (3.86), shows that

$$\begin{aligned} [\![(\bar{\mu},\bar{f}),(\bar{\nu},\bar{g})]\!]^{\hat{}X_0}_{(\Lambda_0,E_0)} &= [\![\Phi^*(\bar{\mu}+\bar{f}\,d_0t),\Phi^*(\bar{\nu}+\bar{g}\,d_0t)]\!]^{\hat{}X_0}_{(\Lambda_0,E_0)}\\ &= \Phi^*[\![\bar{\mu}+\bar{f}\,d_0t,\bar{\nu}+\bar{g}\,d_0t]\!]_{\Pi_0}, \end{aligned} \tag{3.87}$$

$$\widehat{\widetilde{\#}_{(\Lambda_0,E_0)}}^{X_0}(\bar{\mu},\bar{f}) = \widehat{\widetilde{\#}_{(\Lambda_0,E_0)}}^{X_0}(\Phi^*(\bar{\mu}+\bar{f}\,d_0t)) = \#_{\Pi_0}(\bar{\mu}+\bar{f}\,d_0t),$$

for $\bar{\mu},\bar{\nu}$ time-dependent 1-forms on M and $\bar{f},\bar{g}\in C^\infty(M\times\mathbb{R},\mathbb{R})$, where $\Pi_0=e^{-t}(\Lambda_0+\frac{\partial}{\partial t}\wedge E_0)$ is the Poissonization of the Jacobi structure (Λ_0,E_0) and $\Phi^*:T^*(M\times\mathbb{R})\to\bar{A}^*=A^*\times\mathbb{R}$ is the adjoint isomorphism of Φ.

Therefore, $\Phi^*:T^*(M\times\mathbb{R})\to\bar{A}^*=A^*\times\mathbb{R}$ defines an isomorphism between the Lie algebroids $(T^*(M\times\mathbb{R}),[\![\,,\,]\!]_{\Pi_0},\#_{\Pi_0})$ and $(A^*\times\mathbb{R},[\![,]\!]^{\hat{}X_0}_{(\Lambda_0,E_0)},\widehat{\widetilde{\#}_{(\Lambda_0,E_0)}}^{X_0})$ Consequently, using Proposition 3.28, we deduce that, for this particular case, the pair $(\bar{A},\bar{A}^*)$ is a Lie bialgebroid, when we consider on $\bar{A}$ and $\bar{A}^*$ the Lie algebroid structures $([\ ,\ ]^{-\phi_0},\bar{\pi}^{\phi_0})$ and $([\![\,,\,]\!]^{\hat{}X_0}_{(\Lambda_0,E_0)},\widehat{\widetilde{\#}_{(\Lambda_0,E_0)}}^{X_0})$, respectively.
In this Section, we generalize the above result for an arbitrary Jacobi bialgebroid. In fact, we prove

Theorem 3.29 *Let $((A,\phi_0),(A^*,X_0))$ be a Jacobi bialgebroid and (Λ_0,E_0) be the induced Jacobi structure over M. Consider on $\bar{A}$ (resp. $\bar{A}^*$) the Lie algebroid structure $([\![\,,\,]\!]^{-\phi_0},\bar{\rho}^{\phi_0})$ (resp. $([\![\,,\,]\!]_*^{\widehat{X_0}},\widehat{\rho_*}^{X_0})$). Then:*

i) The pair $(\bar{A},\bar{A}^*)$ is a Lie bialgebroid over $M\times\mathbb{R}$.

ii) If Π_0 is the induced Poisson structure on $M\times\mathbb{R}$ then Π_0 is the Poissonization of the Jacobi structure (Λ_0,E_0).

Proof: i) Using (3.75) and (3.83), we obtain that

$$\begin{aligned}\widehat{d_*}^{X_0}[\![\bar{X},\bar{Y}]\!]^{-\phi_0} \;=\;& e^{-t}\Big(d_*^{X_0}[\![\bar{X},\bar{Y}]\!]+d_*^{X_0}(\phi_0(\bar{X})\tfrac{\partial\bar{Y}}{\partial t})\\ &-d_*^{X_0}(\phi_0(\bar{Y})\tfrac{\partial\bar{X}}{\partial t})\quad +X_0\wedge\tfrac{\partial}{\partial t}[\![\bar{X},\bar{Y}]\!]\\ &+X_0\wedge\tfrac{\partial}{\partial t}(\phi_0(\bar{X})\tfrac{\partial\bar{Y}}{\partial t})-X_0\wedge\tfrac{\partial}{\partial t}(\phi_0(\bar{Y})\tfrac{\partial\bar{X}}{\partial t})\Big).\end{aligned}$$

Moreover, applying (3.2), (3.76), (3.77) and (3.78), it follows that

$$\begin{aligned}\widehat{d_*}^{X_0}[\![\bar{X},\bar{Y}]\!]^{-\phi_0} \;=\;& e^{-t}\Big(d_*^{X_0}[\![\bar{X},\bar{Y}]\!]+X_0\wedge[\![\tfrac{\partial\bar{X}}{\partial t},\bar{Y}]\!]+X_0\wedge[\![\bar{X},\tfrac{\partial\bar{Y}}{\partial t}]\!]\\ &+\phi_0(\bar{X})\tfrac{\partial}{\partial t}(d_*^{X_0}\bar{Y})-\phi_0(\bar{Y})\tfrac{\partial}{\partial t}(d_*^{X_0}\bar{X})\\ &-\phi_0(\bar{X})X_0\wedge\tfrac{\partial\bar{Y}}{\partial t}+\phi_0(\bar{Y})X_0\wedge\tfrac{\partial\bar{X}}{\partial t}\\ &+\phi_0(\bar{X})X_0\wedge\tfrac{\partial^2\bar{Y}}{\partial t^2}-\phi_0(\bar{Y})X_0\wedge\tfrac{\partial^2\bar{X}}{\partial t^2}\\ &+\tfrac{\partial}{\partial t}(\phi_0(\bar{X}))X_0\wedge\tfrac{\partial\bar{Y}}{\partial t}-\tfrac{\partial}{\partial t}(\phi_0(\bar{Y}))X_0\wedge\tfrac{\partial\bar{X}}{\partial t}\\ &+d_*^{X_0}(\phi_0(\bar{X}))\wedge\tfrac{\partial\bar{Y}}{\partial t}-d_*^{X_0}(\phi_0(\bar{Y}))\wedge\tfrac{\partial\bar{X}}{\partial t}\Big).\end{aligned}$$

On the other hand, from (3.35), (3.76), (3.81) and (3.83), we have that

$$\begin{aligned}[\![\bar{X},\widehat{d_*}^{X_0}\bar{Y}]\!]^{-\phi_0} \;=\;& e^{-t}\Big([\![\bar{X},d_*^{X_0}\bar{Y}]\!]^{\phi_0}+[\![\bar{X},X_0\wedge\tfrac{\partial\bar{Y}}{\partial t}]\!]^{\phi_0}\\ &+\phi_0(\bar{X})\tfrac{\partial}{\partial t}(d_*^{X_0}\bar{Y})+\phi_0(\bar{X})X_0\wedge\tfrac{\partial^2\bar{Y}}{\partial t^2}\\ &-\tfrac{\partial\bar{X}}{\partial t}\wedge i_{\phi_0}(d_*^{X_0}\bar{Y})+\phi_0(\tfrac{\partial\bar{Y}}{\partial t})\tfrac{\partial\bar{X}}{\partial t}\wedge X_0\Big).\end{aligned}$$

Therefore, using (3.18) and (3.20) and the fact that $\frac{\partial}{\partial t}(\phi_0(\bar{Z}))=\phi_0(\frac{\partial\bar{Z}}{\partial t})$, for

$\bar{Z} \in \Gamma(\bar{A})$,

$$\begin{aligned}[\![\bar{X}, \widehat{d_*}^{X_0}\bar{Y}]\!]^{\phi_0} &= e^{-t}\Big([\![\bar{X}, d_*^{X_0}\bar{Y}]\!]^{\phi_0} + [\![\bar{X}, X_0]\!] \wedge \tfrac{\partial \bar{Y}}{\partial t} + X_0 \wedge [\![\bar{X}, \tfrac{\partial \bar{Y}}{\partial t}]\!] \\ &\quad -\phi_0(\bar{X})X_0 \wedge \tfrac{\partial \bar{Y}}{\partial t} + \phi_0(\bar{X})\tfrac{\partial}{\partial t}(d_*^{X_0}\bar{Y}) + \phi_0(\bar{X})X_0 \wedge \tfrac{\partial^2 \bar{Y}}{\partial t^2} \\ &\quad -\tfrac{\partial \bar{X}}{\partial t} \wedge i_{\phi_0}(d_*^{X_0}\bar{Y}) + \tfrac{\partial}{\partial t}(\phi_0(\bar{Y}))\tfrac{\partial \bar{X}}{\partial t} \wedge X_0\Big).\end{aligned}$$

Finally, from (3.33) and (3.36), we deduce that

$$\widehat{d_*}^{X_0}[\![\bar{X}, \bar{Y}]\!]^{\phi_0} = [\![\bar{X}, \widehat{d_*}^{X_0}\bar{Y}]\!]^{\phi_0} - [\![\bar{Y}, \widehat{d_*}^{X_0}\bar{X}]\!]^{\phi_0}.$$

ii) Using (3.35), (3.79), (3.82) and Theorem 3.13, we obtain that the induced Poisson structure Π_0 on $M \times \mathbb{R}$ is given by

$$\Pi_0(d_0\bar{f}, d_0\bar{g}) = \widehat{d_*}^{X_0}\bar{f} \cdot \bar{d}^{\phi_0}\bar{g} = e^{-t}\Big(d\bar{g} \cdot d_*\bar{f} + \frac{\partial \bar{f}}{\partial t}\rho(X_0)(\bar{g}) - \frac{\partial \bar{g}}{\partial t}\rho(X_0)(\bar{f})\Big),$$

for $\bar{f}, \bar{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$.
On the other hand, using (3.46) and (3.47), we prove that

$$e^{-t}\Big(\Lambda_0 + \frac{\partial}{\partial t} \wedge E_0\Big)(\bar{f}, \bar{g}) = e^{-t}\Big(d\bar{g} \cdot d_*\bar{f} + \frac{\partial \bar{f}}{\partial t}\rho(X_0)(\bar{g}) - \frac{\partial \bar{g}}{\partial t}\rho(X_0)(\bar{f})\Big),$$

for $\bar{f}, \bar{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$. Therefore, Π_0 is the Poissonization of (Λ_0, E_0).

QED

Now, we discuss a converse of Theorem 3.29.

Theorem 3.30 *Let $(A, ([\![\,,\,]\!], \rho), \phi_0)$ be a Jacobi algebroid. Suppose that $(([\![\,,\,]\!]_*, \rho_*), X_0)$ is a Jacobi algebroid structure on A^*. Consider on $\bar{A} = A \times \mathbb{R}$ (resp. $\bar{A}^* = A^* \times \mathbb{R}$) the Lie algebroid structure $([\![\,,\,]\!]^{\phi_0}, \bar{\rho}^{\phi_0})$ (resp. $([\![\,,\,]\!]_*^{X_0}, \widehat{\rho_*}^{X_0})$). If $(\bar{A}, \bar{A}^*)$ is a Lie bialgebroid then the pair $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid.*

Proof: Let $\{\,,\,\}_0^-$ be the induced Poisson bracket on $M \times \mathbb{R}$. Then, from (3.79), (3.82) and Theorem 3.13, it follows that

$$\{\bar{f}, \bar{g}\}_0^- = e^{-t}\Big(d\bar{g} \cdot d_*\bar{f} + \frac{\partial \bar{f}}{\partial t}\rho(X_0)(\bar{g}) + \frac{\partial \bar{g}}{\partial t}\rho_*(\phi_0)(\bar{f}) + \frac{\partial \bar{g}}{\partial t}\frac{\partial \bar{f}}{\partial t}\phi_0(X_0)\Big),$$

for $\bar{f}, \bar{g} \in C^\infty(M \times \mathbb{R}, \mathbb{R})$. Since $\{\,,\,\}_{\bar{0}}$ is skew-symmetric, we have that $\{t, t\}_{\bar{0}} = 0$ which implies that $\phi_0(X_0) = 0$. As a consequence,

$$\{\bar{f}, \bar{g}\}_{\bar{0}} = e^{-t}\Big(d\bar{g} \cdot d_* \bar{f} + \frac{\partial \bar{f}}{\partial t}\rho(X_0)(\bar{g}) + \frac{\partial \bar{g}}{\partial t}\rho_*(\phi_0)(\bar{f})\Big).$$

In particular, if $f \in C^\infty(M, \mathbb{R})$ then, using that $\{f, t\}_{\bar{0}} = -\{t, f\}_{\bar{0}}$, we conclude that $\rho(X_0) = -\rho_*(\phi_0)$.

Now, if $X, Y \in \Gamma(A)$, from (3.75), (3.81) and (3.83), we obtain that

$$\widehat{d_*}^{X_0}[\![X, Y]\!]^{-\phi_0} = e^{-t} d_*^{X_0}[\![X, Y]\!],$$

$$\begin{aligned}
[\![X, \widehat{d_*}^{X_0} Y]\!]^{-\phi_0} - [\![Y, \widehat{d_*}^{X_0} X]\!]^{-\phi_0} &= \bar{\rho}^{\phi_0}(X)(e^{-t}) d_*^{X_0} Y + e^{-t}[\![X, d_*^{X_0} Y]\!]^{-\phi_0} \\
&\quad -\bar{\rho}^{\phi_0}(Y)(e^{-t}) d_*^{X_0} X - e^{-t}[\![Y, d_*^{X_0} X]\!]^{-\phi_0} \\
&= e^{-t}\Big([\![X, d_*^{X_0} Y]\!]^{\phi_0} - [\![Y, d_*^{X_0} X]\!]^{\phi_0}\Big).
\end{aligned}$$

Thus, since $\widehat{d_*}^{X_0}[\![X, Y]\!]^{-\phi_0} = [\![X, \widehat{d_*}^{X_0} Y]\!]^{-\phi_0} - [\![Y, \widehat{d_*}^{X_0} X]\!]^{-\phi_0}$, we deduce (3.33). Finally, if $X \in \Gamma(A)$ then, using the computations in the proof of Theorem 3.29 and the fact that

$$\widehat{d_*}^{X_0}[\![X, e^t Y]\!]^{-\phi_0} = [\![X, \widehat{d_*}^{X_0}(e^t Y)]\!]^{-\phi_0} - [\![e^t Y, \widehat{d_*}^{X_0} X]\!]^{-\phi_0},$$

for all $Y \in \Gamma(A)$, we prove that $\Big([\![X_0, X]\!] + (\mathcal{L}_*^{X_0})_{\phi_0} X\Big) \wedge Y = 0$. But this implies that

$$[\![X_0, X]\!] + (\mathcal{L}_*^{X_0})_{\phi_0} X = 0. \qquad \boxed{QED}$$

In [83] it was proved that if the pair (A, A^*) is a Lie bialgebroid then the pair (A^*, A) is also a Lie bialgebroid. Using this fact, Propositions 3.24 and 3.28 and Theorems 3.29 and 3.30, we conclude that a similar result holds for Jacobi bialgebroids.

Theorem 3.31 *If* $((A, \phi_0), (A^*, X_0))$ *is a Jacobi bialgebroid, so is* $((A^*, X_0), (A, \phi_0))$.

3.6 A characterization of Jacobi bialgebroids

Let $\tau : A \to M$ be a vector bundle over M and $\tau^* : A^* \to M$ be the dual bundle. Then, the cotangent bundle to A, T^*A, is a real vector bundle on A^* with vector bundle projection $r : T^*A \to A^*$ defined by

$$r(\mu_a)(b) = \mu_a(b_a^{\mathbf{v}}), \tag{3.88}$$

for $\mu_a \in T_a^*A$ and $a, b \in A_p = \tau^{-1}(p)$, where $b_a^{\mathbf{v}}$ is the vertical lift of b to $T_aA_p \subseteq T_aA$. Moreover, there exists a canonical isomorphism of vector bundles $R : T^*A^* \to T^*A$ over the identity $Id : A^* \to A^*$. In fact, suppose that (x^i, v_j) are fibred coordinates in A and that (x^i, p_j) are the corresponding coordinates in A^*. Then, we may consider the induced local coordinates $(x^i, v_j; p_{x^i}, p_{v_j})$ (respectively, $(x^i, p_j; p_{x^i}, p_{p_j})$) on T^*A (respectively, T^*A^*). In these coordinates, the local expression of R is

$$R(x^i, p_j; p_{x^i}, p_{p_j}) = (x^i, p_{p_j}; -p_{x^i}, p_j) \tag{3.89}$$

(for more details, see [83]).

Now, assume that $([\![\,,\,]\!], \rho)$ (respectively, $([\![\,,\,]\!]_*, \rho_*)$) is a Lie algebroid structure on A (respectively, A^*) and denote by Π_{A^*} (respectively, Π_A) the associated linear Poisson structure on A^* (respectively, A). Then, one may consider the vector bundle morphism $\Psi = \#_{\Pi_A} \circ R : T^*A^* \to TA$ over the map $\Psi_0 = \rho_* : A^* \to TM$.

Note that the vector bundles $T^*A^* \to A^*$ and $TA \to TM$ are Lie algebroids (see Examples 4 and 9 in Section 1.2.2).

Furthermore, Mackenzie and Xu [83] proved that

Theorem 3.32 *[83] Let $(A, [\![\,,\,]\!], \rho)$ be a Lie algebroid such that the dual bundle A^* to A also admits a Lie algebroid structure $([\![\,,\,]\!]_*, \rho_*)$. Then, (A, A^*) is a Lie bialgebroid if and only if the pair (Ψ, Ψ_0) is a Lie algebroid morphism.*

In this Section, we will obtain the corresponding result in the Jacobi setting. For this purpose, we will introduce the definition of a Jacobi algebroid morphism.

Definition 3.33 *Let (Ψ, Ψ_0) be a vector bundle morphism between two vector bundles $\tau : A \to M$ and $\tau' : A' \to M'$. Moreover, suppose that A (respectively, A') admits a Jacobi algebroid structure $((\llbracket\, ,\, \rrbracket, \rho), \phi_0)$ (respectively, $((\llbracket\, ,\, \rrbracket', \rho'), \phi_0'))$. Then, the pair (Ψ, Ψ_0) is said to be a* Jacobi algebroid morphism *if:*

i) (Ψ, Ψ_0) is a Lie algebroid morphism and

ii) $\tilde{\phi}_0' \circ \Psi = \tilde{\phi}_0$,

where $\tilde{\phi}_0' : A' \to \mathbb{R}$ and $\tilde{\phi}_0 : A \to \mathbb{R}$ are the linear functions induced by the 1-cocycles ϕ_0' and ϕ_0, respectively.

Next, we will prove that one may characterize Jacobi bialgebroids in terms of Jacobi algebroid morphisms.

In fact, assume that $(A, (\llbracket\, ,\, \rrbracket, \rho), \phi_0)$ is a Jacobi algebroid and that the dual bundle A^* to A also admits a Jacobi algebroid structure $((\llbracket\, ,\, \rrbracket_*, \rho_*), X_0)$. Then, the pair $((\llbracket\, ,\, \rrbracket, \rho), \phi_0)$ induces a homogeneous Jacobi structure $(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$ on A^* (see Theorem 2.7) and, thus, the vector bundle $T^*A^* \times \mathbb{R} \to A^*$ is a Lie algebroid with Lie algebroid structure $(\llbracket\, ,\, \rrbracket_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}, \widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})})$. Denote by $d^{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}$ the differential of this Lie algebroid and by $\tilde{X}_0 : A^* \to \mathbb{R}$ the linear function induced by the 1-cocycle $X_0 \in \Gamma(A)$. We have that (see (1.30))

$$\bar{X}_0 = d^{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}(\tilde{X}_0) = (-\#_{\Lambda_{(A^*,\phi_0)}}(d_0\tilde{X}_0), E_{(A^*,\phi_0)}(\tilde{X}_0))$$

is, clearly, a 1-coboundary of the Lie algebroid $(T^*A^* \times \mathbb{R}, \llbracket\, ,\, \rrbracket_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}, \widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})})$. In particular, this implies that:

> *The pair $((\llbracket\, ,\, \rrbracket_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}, \widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}), \bar{X}_0)$ is a Jacobi algebroid structure on the vector bundle $T^*A^* \times \mathbb{R} \to A^*$.*

Now, we consider the tangent Lie algebroid $(TA, \llbracket\, ,\, \rrbracket_T, \rho_T)$ and the corresponding linear Poisson structure $\Pi_{(TA)^*}$ on the vector bundle $(TA)^* \to TM$. The complete lift $\tilde{\phi}_0^c : TA \to \mathbb{R}$ of $\tilde{\phi}_0 : A \to \mathbb{R}$ is a linear function in

$TA \to TM$ and, thus, it induces a section of the vector bundle $(TA)^* \to TM$, which we will denote by $\bar{\phi}_0$. Moreover, we may identify the dual bundle to $TA \to TM$ with the vector bundle $TA^* \to TM$ and, under this identification, the Poisson structure $\Pi_{(TA)^*}$ (respectively, the vertical lift $\bar{\phi}_0^{\mathbf{v}} \in \mathfrak{X}((TA)^*)$ of $\bar{\phi}_0 \in \Gamma((TA)^*)$) is just the complete lift $\Pi^c_{A^*}$ of the linear Poisson structure Π_{A^*} on A^* induced by the Lie algebroid $(A, [\![\,,\,]\!], \rho)$ (respectively, the complete lift $(\phi_0^{\mathbf{v}})^c \in \mathfrak{X}(TA^*)$ of $\phi_0^{\mathbf{v}} \in \mathfrak{X}(A^*)$).

On the other hand, using (1.46), we have that

$$(\mathcal{L}_0)_{(\phi_0^{\mathbf{v}})^c}\Pi^c_{A^*} = ((\mathcal{L}_0)_{\phi_0^{\mathbf{v}}}\Pi_{A^*})^c = 0,$$

which implies that

$$(\mathcal{L}_0)_{\bar{\phi}_0^{\mathbf{v}}}\Pi_{(TA)^*} = 0,$$

i.e., $\bar{\phi}_0 : TA \to TM$ is a 1-cocycle of the tangent Lie algebroid $(TA, [\![\,,\,]\!]_T, \rho_T)$. Consequently, we have proved that:

> *The pair $(([\![\,,\,]\!]_T, \rho_T), \bar{\phi}_0)$ is a Jacobi algebroid structure on the vector bundle $TA \to TM$.*

Next, we consider the homogeneous Jacobi structure $(\Lambda_{(A,X_0)}, E_{(A,X_0)})$ on A induced by the Jacobi algebroid structure $(([\![\,,\,]\!]_*, \rho_*), X_0)$ on A^* and the anchor map $\widetilde{\#}_{(\Lambda_{(A,X_0)},E_{(A,X_0)})} : T^*A \times \mathbb{R} \to TA$ of the 1-jet Lie algebroid $T^*A \times \mathbb{R} \to A$, that is,

$$\widetilde{\#}_{(\Lambda_{(A,X_0)},E_{(A,X_0)})}(\mu_a, \lambda) = \#_{\Lambda_{(A,X_0)}}(\mu_a) + \lambda\, E_{(A,X_0)}(a), \tag{3.90}$$

for $(\mu_a, \lambda) \in T_a^*A \times \mathbb{R}$. Furthermore, we will denote by Δ_{A^*} the Liouville vector field of A^* and by $I_{\Delta_{A^*}} : T^*A^* \times \mathbb{R} \to T^*A^* \times \mathbb{R}$ the isomorphism of vector bundles over the identity $Id : A^* \to A^*$ defined by

$$I_{\Delta_{A^*}}(\mu_{a^*}, \lambda) = (\mu_{a^*}, \lambda - \mu_{a^*}(\Delta_{A^*}(a^*))), \tag{3.91}$$

for $(\mu_{a^*}, \lambda) \in T_{a^*}^*A^* \times \mathbb{R}$. In addition, $T^*A \times \mathbb{R}$ is a vector bundle over A^* with bundle projection $\bar{r} : T^*A \times \mathbb{R} \to A^*$ given by

$$\bar{r}(\mu_a, \lambda) = r(\mu_a), \text{ for } (\mu_a, \lambda) \in T_a^*A^* \times \mathbb{R},$$

and the map $(R, -Id) : T^*A^* \times \mathbb{R} \to T^*A \times \mathbb{R}$ defined by

$$(R, -Id)(\mu_{a^*}, \lambda) = (R(\mu_{a^*}), -\lambda), \text{ for } (\mu_{a^*}, \lambda) \in T^*_{a^*}A^* \times \mathbb{R} \tag{3.92}$$

is an isomorphism of vector bundles over the identity $Id : A^* \to A^*$.

Now, we will denote by $\Psi : T^*A^* \times \mathbb{R} \to TA$ the map given by

$$\Psi = \widetilde{\#}_{(\Lambda_{(A,X_0)}, E_{(A,X_0)})} \circ (R, -Id) \circ I_{\Delta_{A^*}} \tag{3.93}$$

and by $\Psi_0 : A^* \to TM$ the anchor map of the Lie algebroid $(A^*, [\![\,,\,]\!]_*, \rho_*)$, that is, $\Psi_0 = \rho_*$.

Then, from (3.88)-(3.93), it follows that the pair (Ψ, Ψ_0) is a vector bundle morphism between the vector bundles $T^*A^* \times \mathbb{R} \to A^*$ and $TA \to TM$. Thus, we have the following commutative diagram.

$$\begin{array}{ccc} T^*A^* \times \mathbb{R} & \xrightarrow{\quad \Psi \quad} & TA \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\quad \Psi_0 \quad} & TM \end{array}$$

The aim of this Section is to prove the following result.

Theorem 3.34 *Let $(A, ([\![\,,\,]\!], \rho), \phi_0)$ be a Jacobi algebroid over M such that the dual bundle $A^* \to M$ also admits a Jacobi algebroid structure $(([\![\,,\,]\!]_*, \rho_*)$, $X_0)$. Then, $((A, \phi_0), (A^*, X_0))$ is a Jacobi bialgebroid if and only if the pair (Ψ, Ψ_0) is a Jacobi algebroid morphism between the Jacobi algebroids $(T^*A^* \times \mathbb{R}, ([\![\,,\,]\!]_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}, \widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}), \bar{X}_0)$ and $(TA, ([\![\,,\,]\!]_T, \rho_T), \bar{\phi}_0)$.*

In the proof of this Theorem, we will use the following notation:

• We will denote by $\Pi_{\bar{A}^*}$ the Poissonization on $\bar{A}^* = A^* \times \mathbb{R}$ of the Jacobi structure $(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$, that is,

$$\Pi_{\bar{A}^*} = e^{-t}\Big(\Lambda_{(A^*,\phi_0)} + \frac{\partial}{\partial t} \wedge E_{(A^*,\phi_0)}\Big). \tag{3.94}$$

As we know, $\Pi_{\bar{A}^*}$ is a linear Poisson structure on the vector bundle $\bar{A}^* \to M \times \mathbb{R}$ and it induces a Lie algebroid structure $([\![\,,\,]\!]^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$ on the vector bundle $A \times \mathbb{R} \to M \times \mathbb{R}$ given by (2.12) (see Example 6 in Section 2.3). Moreover, we may consider the corresponding cotangent Lie algebroid $(T^*(\bar{A}^*), [\![\,,\,]\!]_{\Pi_{\bar{A}^*}}, \#_{\Pi_{\bar{A}^*}})$ associated with the Poisson structure $\Pi_{\bar{A}^*}$.

• We will denote by $\Pi_{\bar{A}}$ the Poissonization on $\bar{A} = A \times \mathbb{R}$ of the Jacobi structure $(\Lambda_{(A,X_0)}, E_{(A,X_0)})$ on A, i.e.,

$$\Pi_{\bar{A}} = e^{-t}\Big(\Lambda_{(A,X_0)} + \frac{\partial}{\partial t} \wedge E_{(A,X_0)}\Big), \tag{3.95}$$

and by $([\![\,,\,]\!]_*^{\hat{}X_0}, \widehat{\rho_*}^{X_0})$ the corresponding Lie algebroid structure on the vector bundle $\bar{A}^* = A^* \times \mathbb{R} \to M \times \mathbb{R}$.

• Since $(\bar{A} = A \times \mathbb{R}, [\![\,,\,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$ is a Lie algebroid over $M \times \mathbb{R}$ we may consider the tangent Lie algebroid $T(\bar{A}) = T(A \times \mathbb{R}) \to T(M \times \mathbb{R})$. We will denote by $([\![\,,\,]\!]_T^{-\phi_0}, \bar{\rho}_T^{\phi_0})$ the Lie algebroid structure on $T(\bar{A}) \to T(M \times \mathbb{R})$. We recall that

$$\bar{\rho}_T^{\phi_0} = \bar{J} \circ (\bar{\rho}^{\phi_0})^T, \tag{3.96}$$

where $\bar{J} : T(T(M \times \mathbb{R})) \to T(T(M \times \mathbb{R}))$ is the natural involution and $(\bar{\rho}^{\phi_0})^T : T(\bar{A}) \to T(T(M \times \mathbb{R}))$ is the tangent map to $\bar{\rho}^{\phi_0} : \bar{A} \to T(M \times \mathbb{R})$ (see, for instance, [83]). Furthermore, if X and Y are sections of the vector bundle $TA \to TM$ and we denote by $\bar{X}$ and $\bar{Y}$ the sections of the vector bundle $T(\bar{A}) \to T(M \times \mathbb{R})$ defined by

$$\bar{X}(v, (t, \dot{t})) = (X(v), (t, \dot{t})), \qquad \bar{Y}(v, (t, \dot{t})) = (Y(v), (t, \dot{t})),$$

for $v \in TM$ and $(t, \dot{t}) \in T\mathbb{R}$, then (see [83])

$$[\![\bar{X}, \bar{Y}]\!]_T^{-\phi_0}(v, (t, \dot{t})) = ([\![X, Y]\!]_T(v), (t, \dot{t})),$$

that is,

$$[\![\bar{X}, \bar{Y}]\!]_T^{-\phi_0} = \overline{[\![X, Y]\!]}_T. \tag{3.97}$$

• The canonical isomorphism between the vector bundles $T^*(\bar{A}^*) \to \bar{A}^*$ and $T^*(\bar{A}) \to \bar{A}^*$ will be denoted by $\bar{R} : T^*(\bar{A}^*) \to T^*(\bar{A})$.

• As we know (see Proposition 3.24), the isomorphism of vector bundles $\Theta^{-1}: \bar{A} \to \bar{A}$ defined by

$$\Theta^{-1}(a,t) = (e^{-t}a, t) \tag{3.98}$$

induces an isomorphism between the Lie algebroids $(\bar{A}, [\![\,,\,]\!]^{\hat{}\phi_0}, \hat{\rho}^{\phi_0})$ and $(\bar{A}, [\![\,,\,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$. Thus, if $\tilde{\Pi}_{\bar{A}^*}$ is the linear Poisson structure on $\bar{A}^*$ induced by the Lie algebroid $(\bar{A}, [\![\,,\,]\!]^{-\phi_0}, \bar{\rho}^{\phi_0})$, the adjoint map $(\Theta^{-1})^*: \bar{A}^* \to \bar{A}^*$ is a Poisson isomorphism between the Poisson manifolds $(\bar{A}^*, \tilde{\Pi}_{\bar{A}^*})$ and $(\bar{A}^*, \Pi_{\bar{A}^*})$. Therefore, the cotangent map to $(\Theta^{-1})^*$, $((\Theta^{-1})^*)^{T^*}: T^*\bar{A}^* \to T^*\bar{A}^*$ defines a Lie algebroid isomorphism between the cotangent Lie algebroids $(T^*\bar{A}^*, [\![\,,\,]\!]_{\Pi_{\bar{A}^*}}, \#_{\Pi_{\bar{A}^*}})$ and $(T^*\bar{A}^*, [\![\,,\,]\!]_{\tilde{\Pi}_{\bar{A}^*}}, \#_{\tilde{\Pi}_{\bar{A}^*}})$, associated with the Poisson manifolds $(\bar{A}^*, \Pi_{\bar{A}^*})$ and $(\bar{A}^*, \tilde{\Pi}_{\bar{A}^*})$, respectively. We will denote by $\bar{I}_{\Delta_{A^*}}$ the isomorphism $((\Theta^{-1})^*)^{T^*}$. Note that the Poisson structure $\tilde{\Pi}_{\bar{A}^*}$ is given by

$$\tilde{\Pi}_{\bar{A}^*} = \Lambda_{(A^*,\phi_0)} + \left(\frac{\partial}{\partial t} + \Delta_{A^*}\right) \wedge E_{(A^*,\phi_0)}. \tag{3.99}$$

On the other hand, if $\cdot$ denotes the scalar multiplication on TA^* when this space is considered as a vector bundle over TM then a direct computation, using (3.98), proves that

$$\bar{I}_{\Delta_{A^*}}(\mu_{a^*} + \lambda\, d_0t_{|t}) = e^{-t}\cdot\mu_{a^*} + (\lambda - \mu_{a^*}(\Delta_{A^*}(a^*)))d_0t_{|t}, \tag{3.100}$$

for $\mu_{a^*} + \lambda\, d_0t_{|t} \in T^*_{(a^*,t)}(\bar{A}^*) = T^*_{(a^*,t)}(A^* \times \mathbb{R})$, where $e^{-t}\cdot\mu_{a^*}$ is the covector in A^* at the point $e^t a^*$ defined by

$$(e^{-t}\cdot\mu_{a^*})(X_{e^t a^*}) = \mu_{a^*}(e^{-t}\cdot X_{e^t a^*}), \tag{3.101}$$

for $X_{e^t a^*} \in T_{e^t a^*}A^*$.

Proof of Theorem 3.34: Using Theorems 3.29, 3.30 and 3.32, we deduce that $((A,\phi_0),(A^*,X_0))$ is a Jacobi bialgebroid if and only if the pair $(\bar{\Psi}, \bar{\Psi}_0)$ is a morphism between the Lie algebroids $(T^*(\bar{A}^*), [\![\,,\,]\!]_{\Pi_{\bar{A}^*}}, \#_{\Pi_{\bar{A}^*}})$ and $(T(\bar{A}), [\![\,,\,]\!]_T^{-\phi_0}, \bar{\rho}_T^{\phi_0})$, $\bar{\Psi}: T^*(\bar{A}^*) \to T(\bar{A})$ (respectively, $\bar{\Psi}_0: \bar{A}^* \to T(M\times\mathbb{R})$) being the map given by

$$\bar{\Psi} = \#_{\Pi_{\bar{A}}} \circ \bar{R} \circ \bar{I}_{\Delta_{A^*}},$$

(respectively, $\bar{\Psi}_0 = \bar{\rho}_*^{X_0}$).

Thus, we must prove that the pair (Ψ, Ψ_0) is a Jacobi algebroid morphism between the Jacobi algebroids $(T^*A^* \times \mathbb{R}, (\llbracket\, , \,\rrbracket_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}, \widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}), \bar{X}_0)$ and $(TA, (\llbracket\, , \,\rrbracket_T, \rho_T), \bar{\phi}_0)$ if and only if the pair $(\bar{\Psi}, \bar{\Psi}_0)$ is a Lie algebroid morphism between the Lie algebroids $(T^*(\bar{A}^*), \llbracket\, , \,\rrbracket_{\Pi_{\bar{A}^*}}, \#_{\Pi_{\bar{A}^*}})$ and $(T(\bar{A}), \llbracket\, , \,\rrbracket_T^{\bar{\phi}_0}, \bar{\rho}_T^{\phi_0})$

For this purpose, we will proceed in several steps.

First step: We will show that the following diagram

$$\begin{array}{ccc} T^*(\bar{A}^*) & \xrightarrow{\quad \bar{\Psi} \quad} & T(\bar{A}) \\ \Big\downarrow {\scriptstyle \#_{\Pi_{\bar{A}^*}}} & & \Big\downarrow {\scriptstyle \bar{\rho}_T^{\phi_0}} \\ T(\bar{A}^*) & \xrightarrow{\quad (\bar{\rho}_*^{X_0})^T \quad} & T(T(M \times \mathbb{R})) \end{array}$$

is commutative if and only if

$$\begin{aligned} &(J \circ \rho^T)\Big\{ i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*}))) - \lambda\, E_{(A,X_0)}(r^*(\mu_{a^*}))\Big\} \\ &\quad = \rho_*^T\Big\{ i_{\mu_{a^*}}(\Lambda_{(A^*,\phi_0)}(a^*)) + (\lambda + \mu_{a^*}(\Delta_{A^*}(a^*)))E_{(A^*,\phi_0)}(a^*)\Big\}, \end{aligned} \tag{3.102}$$

$$\begin{aligned} &\bar{\phi}_0\Big\{ i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*}))) - \lambda\, E_{(A,X_0)}(r^*(\mu_{a^*}))\Big\} \\ &\quad = \Big\{ i_{\mu_{a^*}}(\Lambda_{(A^*,\phi_0)}(a^*)) + (\lambda + \mu_{a^*}(\Delta_{A^*}(a^*)))E_{(A^*,\phi_0)}(a^*)\Big\}(\tilde{X}_0), \end{aligned} \tag{3.103}$$

for $(\mu_{a^*}, \lambda) \in T^*_{a^*}A^* \times \mathbb{R}$, where $J : T(TM) \to T(TM)$ is the natural involution and $r^* : T^*A^* \to A$ is the bundle projection when T^*A^* is considered as a vector bundle over A.

In fact, using (3.89), (3.95), (3.96), (3.100) and (3.101), we deduce that

$$\begin{aligned} &(\bar{\rho}_T^{\phi_0} \circ \#_{\Pi_{\bar{A}}} \circ \bar{R})(\mu_{a^*} + \lambda\, d_0 t_{|t}) \\ &= e^{-t} \cdot \Big\{ (J \circ \rho^T)\Big[i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*}))) - \lambda\, E_{(A,X_0)}(r^*(\mu_{a^*}))\Big]\Big\} \\ &+ \tilde{\phi}_0(r^*(\mu_{a^*}))\frac{\partial}{\partial t}_{|t} + e^{-t}\Big\{ \Big[i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*}))) \\ &- \lambda\, E_{(A,X_0)}(r^*(\mu_{a^*}))\Big](\tilde{\phi}_0)\Big\} \frac{\partial}{\partial t}_{|e^{-t}R(\mu_{a^*})(E_{(A,X_0)}(r^*(\mu_{a^*})))}, \end{aligned} \tag{3.104}$$

for $\mu_{a^*} + \lambda\, d_0 t_{|t} \in T^*_{(a^*,t)}(\bar{A}^*)$, t being the usual coordinate on $\mathbb{R}$ and $(t, \dot{t})$ the induced coordinates on $T\mathbb{R}$.

On the other hand, from (2.12) and (3.94), it follows that

$$\begin{array}{l}(\hat{\rho}_*^{X_0} \circ \#_{\tilde{\Pi}_{\bar{A}^*}})(\mu_{a^*} + \lambda\, d_0 t_{|t}) \\ = e^{-t} \cdot \Big\{ \rho_*^T \Big(i_{\mu_{a^*}}(\Lambda_{(A^*,\phi_0)}(a^*)) + (\lambda + \mu_{a^*}(\Delta_{A^*}(a^*)))E_{(A^*,\phi_0)}(a^*) \\ - \mu_{a^*}(E_{(A^*,\phi_0)}(a^*))\Delta_{A^*}(a^*) \Big) \Big\} \\ + e^{-t} \Big\{ \Big[i_{\mu_{a^*}}(\Lambda_{(A^*,\phi_0)}(a^*)) + (\lambda + \mu_{a^*}(\Delta_{A^*}(a^*)))E_{(A^*,\phi_0)}(a^*) \\ - \mu_{a^*}(E_{(A^*,\phi_0)}(a^*))\Delta_{A^*}(a^*) \Big] (\tilde{X}_0) \\ + \mu_{a^*}(E_{(A^*,\phi_0)}(a^*))\tilde{X}_0(a^*) \Big\} \frac{\partial}{\partial \dot{t}}_{|e^{-t}\tilde{X}_0(a^*)} \\ - \mu_{a^*}(E_{(A^*,\phi_0)}(a^*)) \frac{\partial}{\partial t}_{|t} + e^{-t} \mu_{a^*}(E_{(A^*,\phi_0)}(a^*))\rho_*(a^*)^{\mathbf{v}}_{|e^{-t}\rho_*(a^*)},\end{array} \tag{3.105}$$

where $\rho_*(a^*)^{\mathbf{v}}_{|e^{-t}\rho_*(a^*)}$ is the vertical lift of $\rho_*(a^*)$ to $T(TM)$ at $e^{-t}\rho_*(a^*)$.

Moreover, using (3.88), (3.89) and since $E_{(A^*,\phi_0)} = -\phi_0^{\mathbf{v}}$ and $E_{(A,X_0)} = -X_0^{\mathbf{v}}$, we have that

$$\begin{array}{l}\mu_{a^*}(E_{(A^*,\phi_0)}(a^*)) = -\phi_0(r^*(\mu_{a^*})), \\ R(\mu_{a^*})(E_{(A,X_0)}(r^*(\mu_{a^*}))) = -X_0(a^*).\end{array} \tag{3.106}$$

In addition, it is easy to prove that

$$\begin{array}{l}\rho_*^T(\Delta_{A^*}(a^*)) = \rho_*(a^*)^{\mathbf{v}}_{|\rho_*(a^*)}, \\ \Delta_{A^*}(a^*)(\tilde{X}_0) = \tilde{X}_0(a^*).\end{array} \tag{3.107}$$

Thus, using (3.104)-(3.107), we deduce the result.

Second step: If $\mu \in \Omega^1(A^*)$ and $f \in C^\infty(A^*, \mathbb{R})$, we will denote by $\overline{(\mu, f)}$ the 1-form on $\bar{A}^*$ defined by

$$\overline{(\mu, f)} = e^t(\mu + f\, d_0 t).$$

Note that

$$\begin{array}{rcl}(\bar{\Psi} \circ \overline{(\mu, f)})(a^*, t) & = & i_{R(\mu(a^*))}(\Lambda_{(A,X_0)}(r^*(\mu(a^*)))) \\ & & + (\mu(a^*)(\Delta_{A^*}(a^*)) - f(a^*))E_{(A,X_0)}(r^*(\mu(a^*))) \\ & & - R(\mu(a^*))(E_{(A,X_0)}(r^*(\mu(a^*)))) \frac{\partial}{\partial t}_{|t},\end{array}$$

for $(a^*, t) \in \bar{A}^*$. Thus, we may consider a $\bar{\Psi}$-decomposition of $\overline{(\mu, f)}$ as follows

$$\bar{\Psi} \circ \overline{(\mu, f)} = \sum_i \bar{u}_i (\bar{X}_i \circ \bar{\rho}_*^{X_0}),$$

with $u_i \in C^\infty(A^*, \mathbb{R})$ and X_i a section of the vector bundle $TA \to TM$, where

$$\bar{u}_i(a^*, t) = u_i(a^*), \qquad \bar{X}_i(v, (t, \dot{t})) = (X_i(v), t, \dot{t}),$$

for $(a^*, t) \in \bar{A}^*$ and $(v, (t, \dot{t})) \in TM \times T\mathbb{R} = T(M \times \mathbb{R})$.

Now, we have that $(\bar{\Psi}, \bar{\Psi}_0)$ is a Lie algebroid morphism over $\bar{\rho}_*^{X_0}$ if and only if (3.102) and (3.103) hold and, in addition, for all $(\mu, f), (\nu, g) \in \Omega^1(A^*) \times C^\infty(A^*, \mathbb{R})$

$$\begin{aligned} \bar{\Psi} \circ [\![\overline{(\mu, f)}, \overline{(\nu, g)}]\!]_{\Pi_{\bar{A}^*}} &= \sum_{i,j} \bar{u}_i \bar{v}_j ([\![\bar{X}_i, \bar{Y}_j]\!]_T^{\bar{\phi}_0} \circ \bar{\rho}_*^{X_0}) \\ &\quad + \sum_j (\#_{\Pi_{\bar{A}^*}} \overline{(\mu, f)})(\bar{v}_j)(\bar{Y}_j \circ \bar{\rho}_*^{X_0}) \\ &\quad - \sum_i (\#_{\Pi_{\bar{A}^*}} \overline{(\nu, g)})(\bar{u}_i)(\bar{X}_i \circ \bar{\rho}_*^{X_0}) \end{aligned} \tag{3.108}$$

where

$$\bar{\Psi} \circ \overline{(\mu, f)} = \sum_i \bar{u}_i (\bar{X}_i \circ \bar{\rho}_*^{X_0}), \quad \bar{\Psi} \circ \overline{(\nu, g)} = \sum_j \bar{v}_j (\bar{Y}_j \circ \bar{\rho}_*^{X_0}),$$

and $u_i, v_j \in C^\infty(A^*, \mathbb{R})$ and X_i, Y_j are sections of the vector bundle $TA \to TM$.

Third step: We will prove that (3.102) and (3.103) hold if and only if the following diagram

$$\begin{array}{ccc} T^*A^* \times \mathbb{R} & \xrightarrow{\Psi} & TA \\ \Big\downarrow \widetilde{\#}_{(\Lambda_{(A^*, \phi_0)}, E_{(A^*, \phi_0)})} & & \Big\downarrow \rho^T \\ TA^* & \xrightarrow{\Psi_0^T} & T(TM) \end{array}$$

is commutative and

$$\widetilde{\bar{\phi}}_0 \circ \Psi = \widetilde{\bar{X}}_0.$$

In fact, the diagram is commutative if and only if

$$\begin{aligned}\rho_*^T\Big\{ i_{\mu_{a^*}}(\Lambda_{(A^*,\phi_0)}(a^*)) + \lambda\, E_{(A^*,\phi_0)}(a^*)\Big\} \\ = (J\circ\rho^T)\Big\{ i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*}))) \\ +(\mu_{a^*}(\Delta_{A^*}(a^*)) - \lambda)E_{(A,X_0)}(r^*(\mu_{a^*}))\Big\}\end{aligned} \tag{3.109}$$

for $(\mu_{a^*},\lambda)\in T^*_{a^*}A^*\times\mathbb{R}$.

Now, it is easy to check that conditions (3.102) and (3.109) are equivalent.

On the other hand, using (3.103), it follows that

$$\begin{aligned}\bar\phi_0(i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*})))) \\ = \mu_{a^*}(\#_{\Lambda_{(A^*,\phi_0)}}(d_0\tilde X_0)(a^*)) + \mu_{a^*}(\Delta_{A^*}(a^*))E_{(A^*,\phi_0)}(a^*)(\tilde X_0),\end{aligned} \tag{3.110}$$

$$-\bar\phi_0(E_{(A,X_0)}(r^*(\mu_{a^*}))) = E_{(A^*,\phi_0)}(a^*)(\tilde X_0), \tag{3.111}$$

for all $a^*\in A^*$ and $\mu_{a^*}\in T^*_{a^*}A^*$. From (3.111) and since $E_{(A,X_0)} = -X_0^{\mathbf{v}}$ and $E_{(A^*,\phi_0)} = -\phi_0^{\mathbf{v}}$, we obtain that

$$E_{(A^*,\phi_0)}(\tilde X_0) = 0, \quad E_{(A,X_0)}(\tilde\phi_0) = 0, \quad \phi_0(X_0) = 0. \tag{3.112}$$

Therefore, using (1.29), (3.91), (3.93), (3.110) and (3.112), we conclude that

$$\tilde{\tilde{\bar\phi}}_0\circ\Psi = \tilde{\tilde{\bar X}}_0.$$

Conversely, assume that

$$\tilde{\tilde{\bar\phi}}_0\circ\Psi = \tilde{\tilde{\bar X}}_0.$$

Then, from (1.29), (3.91) and (3.93), we deduce that

$$\begin{aligned}\bar\phi_0(i_{R(\mu_{a^*})}(\Lambda_{(A,X_0)}(r^*(\mu_{a^*})))) + (\mu_{a^*}(\Delta_{A^*}(a^*)) - \lambda)E_{(A,X_0)}(r^*(\mu_{a^*})) \\ = -\mu_{a^*}(\#_{\Lambda_{(A^*,\phi_0)}}(d_0\tilde X_0)(a^*)) + \lambda\, E_{(A^*,\phi_0)}(a^*)(\tilde X_0),\end{aligned} \tag{3.113}$$

for $(\mu_{a^*},\lambda)\in T^*_{a^*}A^*\times\mathbb{R}$.

This implies that

$$-\bar\phi_0(E_{(A,X_0)}(r^*(\mu_{a^*}))) = E_{(A^*,\phi_0)}(a^*)(\tilde X_0),$$

and thus,

$$E_{(A^*,\phi_0)}(\tilde{X}_0) = 0, \quad E_{(A,X_0)}(\tilde{\phi}_0) = 0, \quad \phi_0(X_0) = 0.$$

Consequently, using (3.113), we obtain that (3.103) holds.

Fourth step: We will show that (Ψ, Ψ_0) is a Jacobi algebroid morphism if and only if $(\bar{\Psi}, \bar{\Psi}_0)$ is a Lie algebroid morphism.

If $(\mu, f), (\nu, g) \in \Omega^1(A^*) \times C^\infty(A^*, \mathbb{R})$ are such that

$$\Psi \circ (\mu, f) = \sum_i u_i (X_i \circ \Psi_0), \quad \Psi \circ (\nu, g) = \sum_j v_j (Y_j \circ \Psi_0),$$

with $u_i, v_j \in C^\infty(A^*, \mathbb{R})$ and X_i, Y_j sections of the vector bundle $TA \to TM$, then we obtain that

$$\bar{\Psi} \circ \overline{(\mu, f)} = \sum_i \bar{u}_i (\bar{X}_i \circ \bar{\rho}_*^{X_0}), \quad \bar{\Psi} \circ \overline{(\nu, g)} = \sum_j \bar{v}_j (\bar{Y}_j \circ \bar{\rho}_*^{X_0}),$$

where $\bar{u}_i, \bar{v}_j \in C^\infty(\bar{A}^*, \mathbb{R})$ and $\bar{X}_i, \bar{Y}_j$ are sections of the vector bundle $T\bar{A} \to T(M \times \mathbb{R})$ defined by

$$\begin{aligned} &\bar{u}_i(a^*, t) = u_i(a^*), \qquad \bar{v}_j(a^*, t) = v_j(a^*), \\ &\bar{X}_i(v, (t, \dot{t})) = (X_i(v), t, \dot{t}), \qquad \bar{Y}_j(v, (t, \dot{t})) = (Y_j(v), t, \dot{t}), \end{aligned}$$

for $(a^*, t) \in \bar{A}^*$ and $(v, (t, \dot{t})) \in TM \times T\mathbb{R} = T(M \times \mathbb{R})$.
In addition, we have that

$$\begin{aligned} (\#_{\Pi_{\bar{A}^*}} \overline{(\mu, f)})(\bar{v}_j) &= \overline{\widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}(\mu, f)(v_j)} \\ (\#_{\Pi_{\bar{A}^*}} \overline{(\nu, g)})(\bar{u}_i) &= \overline{\widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}(\nu, g)(u_i)}. \end{aligned}$$

Using these facts, (1.33) and (3.97), we conclude that (3.108) holds if and only if

$$\begin{aligned} \Psi \circ [\![(\mu, f), (\nu, g)]\!]_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})} &= \sum_{i,j} u_i v_j ([\![X_i, Y_j]\!]_T \circ \rho_*) \\ &+ \sum_j (\widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}(\mu, f))(v_j)(Y_j \circ \rho_*) \\ &- \sum_i (\widetilde{\#}_{(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})}(\nu, g))(u_i)(X_i \circ \rho_*). \end{aligned}$$

This ends the proof of the result. $\boxed{QED}$

Suppose that $(A, [\![\, , \,]\!], \rho)$ is a Lie algebroid such that the dual bundle to A, A^*, admits a Lie algebroid structure $([\![\, , \,]\!]_*, \rho_*)$. Then, the morphism $\Psi : T^*A^* \times \mathbb{R} \to TA$ is given by (see (3.93))

$$\Psi(\mu_{a^*}, \lambda) = \#_{\Pi_A}(R(\mu_{a^*})),$$

for $(\mu_{a^*}, \lambda) \in T^*_{a^*}A^* \times \mathbb{R}$, Π_A being the linear Poisson structure on A induced by the Lie algebroid $(A^*, [\![\, , \,]\!]_*, \rho_*)$. On the other hand, since $\phi_0 = 0$ and $X_0 = 0$,

$$\Lambda_{(A^*,\phi_0)} = \Lambda_{(A^*,0)} = \Pi_{A^*}, \qquad E_{(A^*,\phi_0)} = E_{(A^*,0)} = 0,$$
$$\bar{\phi}_0 = 0, \qquad \bar{X}_0 = 0,$$

where Π_{A^*} is the linear Poisson structure on A^* induced by the Lie algebroid A. In addition,

$$[\![(\mu, f), (\nu, g)]\!]_{(\Lambda_{(A^*,0)}, E_{(A^*,0)})} = ([\![\mu, \nu]\!]_{\Pi_{A^*}}, -\Pi_{A^*}(\mu, \nu)),$$
$$\widetilde{\#}_{(\Lambda_{(A^*,0)}, E_{(A^*,0)})}(\mu, f) = \#_{\Pi_{A^*}}(\mu),$$

for $(\mu, f), (\nu, g) \in \Omega^1(A^*) \times C^\infty(A^*, \mathbb{R})$.

Thus, using Theorem 3.34, we directly deduce Theorem 3.32.

CHAPTER 4

Jacobi bialgebras

In this Chapter, we study Jacobi bialgebroids over a single point, that is, Jacobi bialgebras. We propose a method generalizing the Yang-Baxter equation method to obtain Jacobi bialgebras and give some examples of Jacobi bialgebras. Finally, we discuss compact Jacobi bialgebras.

4.1 Algebraic Jacobi structures

In this Section, we will deal with an algebraic version of the concept of Jacobi structure.

Definition 4.1 *Let* $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ *be a real Lie algebra of finite dimension. An* algebraic Jacobi structure *on* $\mathfrak{g}$ *is a pair* (r, X_0')*, with* $r \in \wedge^2\mathfrak{g}$ *and* $X_0' \in \mathfrak{g}$ *satisfying*

$$[r, r]_{\mathfrak{g}} = 2X_0' \wedge r, \qquad [X_0', r]_{\mathfrak{g}} = 0,$$

where $[\,,\,]_{\mathfrak{g}}$ *is the algebraic Schouten bracket.*

Note that the algebraic Poisson structures on $\mathfrak{g}$ or, in other words, the solutions of the classical Yang-Baxter equation on $\mathfrak{g}$ are just the algebraic Jacobi structures (r, X_0) such that X_0' is zero.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. If $s \in \wedge^k\mathfrak{g}$ then we will denote by $\overleftarrow{s}$ the left-invariant k-vector field on G defined by $\overleftarrow{s}(g) = (L_g)_*(s)$, for all $g \in G$. Since $[\overleftarrow{s}, \overleftarrow{t}] = \overleftarrow{[s,t]_{\mathfrak{g}}}$, for $s, t \in \wedge^*\mathfrak{g}$, the pair (r, X_0') is an algebraic Jacobi structure on $\mathfrak{g}$ if and only if $(\overleftarrow{r}, \overleftarrow{X'}_0)$ is a left invariant Jacobi structure on G.

Examples 4.2 **1.-** *Contact Lie algebras*
Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of odd dimension $2k+1$. We say that $\eta \in \mathfrak{g}^*$ is an *algebraic contact 1-form* on $\mathfrak{g}$ if $\eta \wedge (d\eta)^k = \eta \wedge d\eta \wedge \overset{(k}{\ldots} \wedge d\eta \neq 0$, where d is the Chevalley-Eilenberg differential of $\mathfrak{g}$ (see [25]). In such a case, $(\mathfrak{g}, \eta)$ is termed a *contact Lie algebra*. If $(\mathfrak{g}, \eta)$ is a contact Lie algebra, we define $r \in \wedge^2\mathfrak{g}$ and $X_0' \in \mathfrak{g}$ as follows

$$r(\mu, \nu) = d\eta(\flat_\eta^{-1}(\mu), \flat_\eta^{-1}(\nu)), \qquad X_0' = \flat_\eta^{-1}(\eta), \tag{4.1}$$

for $\mu, \nu \in \mathfrak{g}^*$, where $\flat_\eta \colon \mathfrak{g} \to \mathfrak{g}^*$ is the isomorphism of vector spaces given by

$$\flat_\eta(X) = i_X(d\eta) + \eta(X)\eta, \tag{4.2}$$

for $X \in \mathfrak{g}$. The vector X_0' is the *Reeb vector* of $\mathfrak{g}$ and it is characterized by the relations

$$i_{X_0'}(d\eta) = 0, \qquad \eta(X_0') = 1. \tag{4.3}$$

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the left invariant 1-form $\overleftarrow{\eta}$ on G satisfying $\overleftarrow{\eta}(\mathfrak{e}) = \eta$ is a contact 1-form. Moreover, the pair $(\overleftarrow{r}, \overleftarrow{X'}_0)$ is just the Jacobi structure on G associated with $\overleftarrow{\eta}$ (see, for instance, [24, 39, 74]; see also Section 1.2.2). Therefore, we deduce that (r, X_0') is an algebraic Jacobi structure on $\mathfrak{g}$.

Using (4.1), (4.2) and (4.3) (see also Remark 1.1), we find that $\#_r(\mu) = -\flat_\eta^{-1}(\mu) + \mu(X_0')X_0'$, for $\mu \in \mathfrak{g}^*$, where $\#_r : \mathfrak{g}^* \to \mathfrak{g}$ is the corresponding linear map induced by r.

2.- *Locally conformal symplectic Lie algebras*
Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of even dimension $2k$. An *algebraic locally conformal symplectic (l.c.s.) structure* on $\mathfrak{g}$ is a pair (Ω, ω), where $\Omega \in \wedge^2\mathfrak{g}^*$,

$\omega \in \mathfrak{g}^*$ and

$$\Omega^k = \Omega \wedge \overset{(k}{\ldots} \wedge \Omega \neq 0, \quad d\Omega = \omega \wedge \Omega, \qquad d\omega = 0. \tag{4.4}$$

The 1-form ω is the *Lee 1-form* of the l.c.s. structure.

If (Ω, ω) is an algebraic l.c.s. structure on $\mathfrak{g}$, one can define $r \in \wedge^2\mathfrak{g}$ and $X'_0 \in \mathfrak{g}$ by

$$r(\mu, \nu) = \Omega(\flat_\Omega^{-1}(\mu), \flat_\Omega^{-1}(\nu)), \qquad X'_0 = \flat_\Omega^{-1}(\omega), \tag{4.5}$$

for $\mu, \nu \in \mathfrak{g}^*$, $\flat_\Omega \colon \mathfrak{g} \to \mathfrak{g}^*$ being the isomorphism of vector spaces given by

$$\flat_\Omega(X) = i_X\Omega, \tag{4.6}$$

for $X \in \mathfrak{g}$. If G is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the left invariant 2-form $\overleftarrow{\Omega}$ defines a locally conformal symplectic structure on G. Furthermore, the pair $(\overleftarrow{r}, \overleftarrow{X'}_0)$ is just the Jacobi structure on G associated with $\overleftarrow{\Omega}$ (see, for instance, [39, 57]; see also Section 1.2.2). Consequently, we obtain that (r, X'_0) is an algebraic Jacobi structure on $\mathfrak{g}$.

In this case, using (4.5) and (4.6) (see also Remark 1.1), it follows that $\#_r(\mu) = -\flat_\Omega^{-1}(\mu)$, for $\mu \in \mathfrak{g}^*$. In particular, $\#_r \colon \mathfrak{g}^* \to \mathfrak{g}$ is a linear isomorphism.

It is clear that a real Lie algebra $\mathfrak{g}$ is symplectic in the sense of [76] if and only if $\mathfrak{g}$ is l.c.s. and the Lee 1-form is zero. Moreover, if $\mathfrak{g}$ is a symplectic Lie algebra then the 2-vector $r \in \wedge^2\mathfrak{g}$ given by (4.5) is a solution of the classical Yang-Baxter equation on $\mathfrak{g}$.

Now, we introduce the following definition.

Definition 4.3 *Let* $(\mathfrak{g}, [\,,\,]_\mathfrak{g})$ *be a real Lie algebra of dimension* n *and* (r, X'_0) *be an algebraic Jacobi structure on* $\mathfrak{g}$*. The rank of* (r, X'_0) *is the dimension of the subspace* $\#_r(\mathfrak{g}^*) + < X'_0 > \subseteq \mathfrak{g}$.

Equivalently, the rank of (r, X'_0) is $2k \leq n$ (respectively, $2k + 1 \leq n$) if the rank of r is $2k$ and $X'_0 \wedge r^k = X'_0 \wedge r \wedge \overset{(k}{\ldots} \wedge r = 0$ (respectively, $X'_0 \wedge r^k \neq 0$).

If G is a connected Lie group with Lie algebra $\mathfrak{g}$ then it is clear that the rank of an algebraic Jacobi structure (r, X_0') on $\mathfrak{g}$ is just the rank of the Jacobi structure $(\overleftarrow{r}, \overleftarrow{X'}_0)$ on G. Thus, the rank of a contact Lie algebra (respectively, l.c.s. Lie algebra) of dimension $2k+1$ (respectively, $2k$) is $2k+1$ (respectively, $2k$). Conversely, using some well-known results about transitive Jacobi manifolds (see [24, 39, 57]; see also Remark 1.2), one may prove that if (r, X_0') is an algebraic Jacobi structure of rank $2k+1$ (respectively, of rank $2k$) on a Lie algebra $\mathfrak{g}$ of dimension $2k+1$ (respectively, of dimension $2k$) then the structure (r, X_0') is associated with an algebraic contact structure (respectively, an algebraic l.c.s. structure) on $\mathfrak{g}$. Moreover,

Proposition 4.4 *Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of dimension n and (r, X_0') be an algebraic Jacobi structure on $\mathfrak{g}$ of rank $m \leq n$. Then, there exists an m-dimensional Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $r \in \wedge^2\mathfrak{h}$, $X_0' \in \mathfrak{h}$, the pair (r, X_0') defines an algebraic Jacobi structure on $\mathfrak{h}$ and:*

i) If m is odd, the structure (r, X_0') is associated with an algebraic contact structure on $\mathfrak{h}$.

ii) If m is even, the structure (r, X_0') is associated with an algebraic l.c.s. structure on $\mathfrak{h}$.

Proof: Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $(\overleftarrow{r}, \overleftarrow{X'}_0)$ be the corresponding left invariant Jacobi structure on G. Denote by $\mathcal{F}$ the characteristic foliation on G associated with the Jacobi structure $(\overleftarrow{r}, \overleftarrow{X'}_0)$, that is (see Section 1.1.3), for every $g \in G$, $\mathcal{F}_g$ is the subspace of T_gG defined by $\mathcal{F}_g = (\#_{\overleftarrow{r}})(T_g^*G) + < \overleftarrow{X'}_0(g) >$. It is clear that

$$\overleftarrow{r}(g) \in \wedge^2\mathcal{F}_g, \quad \mathcal{F}_g = (L_g)_*(\mathcal{F}_{\mathfrak{e}}), \quad dim\,\mathcal{F}_g = dim\,\mathcal{F}_{\mathfrak{e}} = m,$$

for all $g \in G$. Thus, $\mathfrak{h} = \mathcal{F}_{\mathfrak{e}}$ is an m-dimensional Lie subalgebra of $\mathfrak{g}$ satisfying the conclusions of the proposition. *QED*

In [25], Diatta proved that if G is a Lie group which admits a left invariant contact structure and a bi-invariant semi-Riemannian metric, then G is semisimple and thus, from Theorem 5 in [5], he deduced that G is locally isomorphic to $SL(2,\mathbb{R})$ or to $SU(2)$. Therefore, if $\mathfrak{h}$ is a compact Lie algebra endowed with an algebraic contact structure, then $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$. Here, we will give a direct proof of this last assertion, and we will describe all the algebraic contact structures on $\mathfrak{su}(2)$.

Proposition 4.5 *Let $\mathfrak{h}$ be a compact Lie algebra of dimension $2k+1$, with $k \geq 1$. Suppose that (r, X_0') is an algebraic Jacobi structure on $\mathfrak{h}$ which is associated with an algebraic contact structure. Then, $k=1$, $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$ and*

$$r = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \qquad X_0' = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

where $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0,0,0)\}$ and $\{e_1, e_2, e_3\}$ is a basis of $\mathfrak{h}$ such that

$$[e_1, e_2]_{\mathfrak{h}} = e_3, \quad [e_3, e_1]_{\mathfrak{h}} = e_2, \quad [e_2, e_3]_{\mathfrak{h}} = e_1.$$

Proof: Let η be the algebraic contact 1-form on $\mathfrak{h}$ associated with the algebraic Jacobi structure (r, X_0') (see (4.1)). We can consider an *ad*-invariant scalar product $\langle\, , \rangle\colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ on $\mathfrak{h}$ and the vector $X_\eta \in \mathfrak{h}$ characterized by the relation

$$\eta(X) = \langle X, X_\eta \rangle, \text{ for } X \in \mathfrak{h}. \tag{4.7}$$

If d is the Chevalley-Eilenberg differential on $\mathfrak{h}$ then, using (4.7) and the fact that $\langle\, , \rangle$ is an *ad*-invariant scalar product, we see that $i_{X_\eta}(d\eta) = 0$. This implies that

$$Ker(d\eta) =< X_0' >=< X_\eta > . \tag{4.8}$$

Next, we will prove that the rank of $\mathfrak{h}$, as a compact Lie algebra, is 1. Assume that there exists $Y \in \mathfrak{h}$ such that $[X_\eta, Y]_{\mathfrak{h}} = 0$. From (4.7), we obtain that $(i_Y d\eta)(X) = -\langle X_\eta, [Y, X]_{\mathfrak{h}} \rangle = 0$, for all $X \in \mathfrak{g}$. Thus, using (4.8), we deduce that X_η and Y are linearly dependent.

Therefore, $< X_\eta >$ is a maximal abelian subspace of $\mathfrak{h}$. This implies that the rank of $\mathfrak{h}$ is 1 and $\mathfrak{h}$ is isomorphic to $\mathfrak{su}(2)$.

Let η be an arbitrary 1-form on $\mathfrak{h}$, $\eta \neq 0$, then η is an algebraic contact 1-form. If $\eta = \mu_1 e^1 + \mu_2 e^2 + \mu_3 e^3$, where $\{e^1, e^2, e^3\}$ denotes the dual basis of $\{e_1, e_2, e_3\}$, the algebraic Jacobi structure (r, X_0') associated with η is given by (see (4.1))

$$r = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \quad X_0' = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3)$$

with $\lambda^i = -\frac{\mu_i}{(\mu_1^2+\mu_2^2+\mu_3^2)}$, for $i \in \{1, 2, 3\}$. QED

4.2 Coboundary Jacobi bialgebras

In this Section, we will deal with a particular class of Jacobi bialgebroids over a point.

Definition 4.6 *A* Jacobi bialgebra *is a Jacobi bialgebroid over a point, that is, a pair* $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, *where* $(\mathfrak{g}, [\, , \,]_{\mathfrak{g}})$ *is a real Lie algebra of finite dimension such that the dual space* $\mathfrak{g}^*$ *is also a Lie algebra with Lie bracket* $[\, , \,]_{\mathfrak{g}^*}$, $X_0 \in \mathfrak{g}$ *and* $\phi_0 \in \mathfrak{g}^*$ *are 1-cocycles on* $\mathfrak{g}^*$ *and* $\mathfrak{g}$, *respectively, and*

$$d_*^{X_0}[X, Y]_{\mathfrak{g}} = [X, d_*^{X_0} Y]_{\mathfrak{g}}^{\phi_0} - [Y, d_*^{X_0} X]_{\mathfrak{g}}^{\phi_0}, \tag{4.9}$$

$$\phi_0(X_0) = 0, \tag{4.10}$$

$$i_{\phi_0}(d_* X) + [X_0, X]_{\mathfrak{g}} = 0, \tag{4.11}$$

for all $X, Y \in \mathfrak{g}$. *Here,* d_* *being the Chevalley-Eilenberg differential of* $(\mathfrak{g}^*, [\, , \,]_{\mathfrak{g}^*})$ *(acting on* $\mathfrak{g} = \wedge^1 \mathfrak{g} \subset \wedge^* \mathfrak{g}$*),* $d_*^{X_0}$ *is the* X_0*-differential of* $(\mathfrak{g}^*, [\, , \,]_{\mathfrak{g}^*}, X_0)$ *and* $[\, , \,]_{\mathfrak{g}}^{\phi_0}$ *is the algebraic* ϕ_0*-Schouten bracket of* $\mathfrak{g}$.

Remark 4.7 In the particular case when $\phi_0 = 0$ and $X_0 = 0$, we recover the concept of a *Lie bialgebra* [27], that is, a pair of Lie algebras in duality $(\mathfrak{g}, \mathfrak{g}^*)$ such that $d_*[X, Y]_{\mathfrak{g}} = [X, d_* Y]_{\mathfrak{g}} - [Y, d_* X]_{\mathfrak{g}}$, for $X, Y \in \mathfrak{g}$ (see [61, 83]).

Let $\mathfrak{g}$ be a Lie algebra, $\phi_0 \in \mathfrak{g}^*$ be a 1-cocycle and $c \in \mathbb{R}$. We can introduce the representation $ad_{(\phi_0,c)} \colon \mathfrak{g} \times \wedge^k \mathfrak{g} \to \wedge^k \mathfrak{g}$ of $\mathfrak{g}$ on $\wedge^k \mathfrak{g}$ given by

$$ad_{(\phi_0,c)}(X)(s) = [X, s]_{\mathfrak{g}} - (k-c)\phi_0(X)s = ad(X)(s) - (k-c)\phi_0(X)s, \tag{4.12}$$

for $X \in \mathfrak{g}$ and $s \in \wedge^k \mathfrak{g}$, where $[\,,\,]_{\mathfrak{g}}$ is the algebraic Schouten bracket. It is clear that if $c = 1$ then (see (3.22))

$$ad_{(\phi_0,1)}(X)(s) = [X, s]_{\mathfrak{g}}^{\phi_0}. \tag{4.13}$$

Now, assume that $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra. Then, from (4.9) and (4.13), we deduce that $d_*^{X_0}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the representation $ad_{(\phi_0,1)} \colon \mathfrak{g} \times \wedge^2 \mathfrak{g} \to \wedge^2 \mathfrak{g}$. Next, we will propose a method to obtain Jacobi bialgebras such that $d_*^{X_0}$ is a 1-coboundary (i.e., there exists $r \in \wedge^2 \mathfrak{g}$ satisfying that $d_*^{X_0} X = ad_{(\phi_0,1)}(X)(r)$, for $X \in \mathfrak{g}$). It is a generalization of the well-known Yang-Baxter equation method to obtain Lie bialgebras (see, for instance, [110]).

Theorem 4.8 *Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle and that $r \in \wedge^2 \mathfrak{g}$ and $X_0 \in \mathfrak{g}$ are such that*

$$[r, r]_{\mathfrak{g}} + 2X_0 \wedge r \text{ is } ad_{(\phi_0,1)}\text{-invariant}, \tag{4.14}$$

$$[X_0, r]_{\mathfrak{g}} = 0, \tag{4.15}$$

$$i_{\phi_0}(r) + X_0 \text{ is } ad_{(\phi_0,0)}\text{-invariant}. \tag{4.16}$$

If $[\,,\,]_{\mathfrak{g}^*}$ is the bracket on $\mathfrak{g}^*$ given by

$$[\mu, \nu]_{\mathfrak{g}^*} = coad_{\#_r(\mu)}\nu - coad_{\#_r(\nu)}\mu - r(\mu, \nu)\phi_0 + i_{X_0}(\mu \wedge \nu), \tag{4.17}$$

for $\mu, \nu \in \mathfrak{g}^*$, where $coad \colon \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the coadjoint representation of $\mathfrak{g}$ over $\mathfrak{g}^*$, that is, $(coad_X \mu)(Y) = -\mu[X, Y]_{\mathfrak{g}}$, for $X, Y \in \mathfrak{g}$, then $(\mathfrak{g}^*, [\,,\,]_{\mathfrak{g}^*})$ is a Lie algebra and the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra.

Proof: From (1.20), (4.12), (4.14) and (4.16), it follows that

$$\begin{aligned} &[[r, r]_{\mathfrak{g}} + 2X_0 \wedge r, s]_{\mathfrak{g}} + 2([r, r]_{\mathfrak{g}} + 2X_0 \wedge r) \wedge i_{\phi_0}(s) = 0, \\ &[i_{\phi_0}(r) + X_0, s]_{\mathfrak{g}} + (i_{\phi_0}(r) + X_0) \wedge i_{\phi_0}(s) = 0, \end{aligned} \tag{4.18}$$

for all $s \in \wedge^k \mathfrak{g}$. Moreover, using (4.16), we obtain that

$$\phi_0(X_0) = 0. \tag{4.19}$$

Now, we define the $\mathbb{R}$-linear map $d_* : \wedge^*\mathfrak{g} \to \wedge^{*+1}\mathfrak{g}$ by

$$d_* s = -[r, s]_{\mathfrak{g}} + r \wedge (i_{\phi_0} s) - kX_0 \wedge s, \tag{4.20}$$

for $s \in \wedge^k\mathfrak{g}$.

From (1.20), we deduce that d_* is a derivation with respect to $(\oplus_k \wedge^k \mathfrak{g}, \wedge)$. Furthermore, using (1.20), (3.23), (4.15), (4.18) and (4.19), we conclude that $d_*^2 = 0$. Thus, the equation

$$[\mu, \nu]_{\mathfrak{g}^*}(X) = -d_*X(\mu, \nu),$$

for $\mu, \nu \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, defines the Lie bracket $[\,,\,]_{\mathfrak{g}^*}$ of a Lie algebra structure on $\mathfrak{g}^*$.

A simple computation, using (4.20) and the fact that

$$(coad_{\#_r(\mu)}\nu - coad_{\#_r(\nu)}\mu)(X) = [r, X]_{\mathfrak{g}}(\mu, \nu),$$

for $\mu, \nu \in \mathfrak{g}^*$ and $X \in \mathfrak{g}$, shows that $[\,,\,]_{\mathfrak{g}^*}$ is given by (4.17).

Moreover, from (4.15), (4.19) and (4.20), we get that X_0 is a 1-cocycle of $(\mathfrak{g}^*, [\,,\,]_{\mathfrak{g}^*})$.

On the other hand, using (3.23), (4.12), (4.16), (4.19) and (4.20), we have that

$$i_{\phi_0}(d_*X) + [X_0, X]_{\mathfrak{g}} = 0, \text{ for } X \in \mathfrak{g}.$$

Furthermore, from (4.20), we deduce that

$$d_*^{X_0} X = -[X, r]_{\mathfrak{g}}^{\phi_0}, \text{ for } X \in \mathfrak{g}.$$

Thus, using (3.21), we conclude that (4.9) holds. Therefore, the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra. *QED*

Remark 4.9 *i)* From (4.20) we deduce that

$$d_*r = -[r, r]_{\mathfrak{g}} - 2X_0 \wedge r + (i_{\phi_0} r) \wedge r. \tag{4.21}$$

ii) If $X \in \mathfrak{g}$, it follows that (see (4.17) and (4.20))

$$\begin{array}{rcl} [\mu,\nu]_{\mathfrak{g}^*}(X) & = & [X,r]_{\mathfrak{g}}(\mu,\nu) - r(\mu,\nu)\phi_0(X) \\ & & +\mu(X_0)\nu(X) - \nu(X_0)\mu(X). \end{array} \tag{4.22}$$

Remark 4.10 Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle and that $r \in \wedge^2\mathfrak{g}$ satisfies the following relations

$$[r,r]_{\mathfrak{g}} - 2i_{\phi_0}(r) \wedge r = 0, \qquad [i_{\phi_0}(r), r]_{\mathfrak{g}} = 0,$$

that is, the pair $(r, i_{\phi_0}(r))$ is an algebraic Jacobi structure (see Definition 4.1). Then, r is a ϕ_0-canonical section on $\mathfrak{g}$ and, using Theorem 3.20, we deduce that $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,-i_{\phi_0}(r)))$ is a triangular Jacobi bialgebroid. In particular, this implies that $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,-i_{\phi_0}(r)))$ is a Jacobi bialgebra.

Now, using Theorem 4.8, we have

Corollary 4.11 *Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle and that $r \in \wedge^2\mathfrak{g}$ is such that $(r, i_{\phi_0}(r))$ is an algebraic Jacobi structure on $\mathfrak{g}$. If $[\,,\,]_{\mathfrak{g}^*}$ is the Lie bracket on $\mathfrak{g}^*$ given by (4.17), then $(\mathfrak{g}^*, [\,,\,]_{\mathfrak{g}^*})$ is a Lie algebra and the pair $((\mathfrak{g},\phi_0),\ (\mathfrak{g}^*,-i_{\phi_0}(r)))$ is a Jacobi bialgebra. Moreover, the linear map $\#_r\colon \mathfrak{g}^* \to \mathfrak{g}$ is a Lie algebra homomorphism.*

Proof: From Definition 4.1 and Theorem 4.8, we deduce that the pair $((\mathfrak{g},\phi_0),$ $(\mathfrak{g}^*,-i_{\phi_0}(r)))$ is a Jacobi bialgebra. On the other hand, if $\mu,\nu,\gamma \in \mathfrak{g}^*$ then the equality $[r,r]_{\mathfrak{g}}(\mu,\nu,\gamma) = 2(i_{\phi_0}(r)\wedge r)(\mu,\nu,\gamma)$ implies that $-[\mu,\nu]_{\mathfrak{g}^*}\,(\#_r(\gamma)) = \gamma[\#_r(\mu),\#_r(\nu)]_{\mathfrak{g}}$ and therefore

$$\#_r([\mu,\nu]_{\mathfrak{g}^*}) = [\#_r(\mu),\#_r(\nu)]_{\mathfrak{g}}.$$

QED

Remark 4.12 Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a real Lie algebra of finite dimension. Assume that (Ω,ω) is an algebraic locally conformal symplectic (l.c.s.) structure on $\mathfrak{g}$ and denote by (r, X_0) the corresponding algebraic Jacobi structure on $\mathfrak{g}$ (see Examples 4.2). Then, using Corollary 4.11 and the fact that $X_0 = -\#_r(\omega)$, we deduce that the pair $((\mathfrak{g},-\omega),(\mathfrak{g}^*,-X_0))$ is a Jacobi bialgebra. Furthermore, since $\#_r\colon \mathfrak{g}^* \to \mathfrak{g}$ is a linear isomorphism, it follows that $\mathfrak{g}^*$ is isomorphic, as a Lie algebra, to $\mathfrak{g}$.

4.3 Examples of Jacobi bialgebras

First, we will give some examples of Jacobi bialgebras which are obtained using Theorem 4.8 and Corollary 4.11.

4.3.1 Jacobi bialgebras from contact Lie algebras

Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a Lie algebra endowed with an algebraic contact 1-form η and let X_0' be the Reeb vector of $\mathfrak{g}$ (see Examples 4.2). If $\mathcal{Z}(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $X \in \mathcal{Z}(\mathfrak{g})$ then it is clear that $i_X(d\eta) = 0$. This implies that $X \in < X_0' >$. Thus, $\mathcal{Z}(\mathfrak{g}) \subseteq < X_0' >$ (see [25]). Therefore, we have two possibilities: $\mathcal{Z}(\mathfrak{g}) = \{0\}$ or $\mathcal{Z}(\mathfrak{g}) =< X_0' >$.

If $\mathcal{Z}(\mathfrak{g}) =< X_0' >$ then Diatta [25] proved that $\mathfrak{g}$ is the central extension of a symplectic Lie algebra $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ by $\mathbb{R}$ via the 2-cocycle Ω, Ω being the algebraic symplectic structure on $\mathfrak{h}$. Conversely, if $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ is a symplectic Lie algebra, with algebraic symplectic 2-form Ω, and on the direct product $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ we consider the Lie bracket $[\,,\,]_{\mathfrak{g}}$ given by

$$[(X,\lambda),(Y,\mu)]_{\mathfrak{g}} = ([X,Y]_{\mathfrak{h}}, -\Omega(X,Y)), \text{ for } (X,\lambda),(Y,\mu) \in \mathfrak{g}, \tag{4.23}$$

then $\eta = (0,1) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$ is an algebraic contact 1-form on $\mathfrak{g}$. Moreover, since $X_0' = (0,1) \in \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}$, we deduce that $\mathcal{Z}(\mathfrak{g}) =< X_0' >$ (see [25]).

Now, suppose that r is the algebraic Poisson 2-vector on $\mathfrak{h}$ associated with the algebraic symplectic structure Ω and denote by X_0 the vector defined by $X_0 = -X_0'$. Then, the pair $(r, -X_0)$ is the algebraic Jacobi structure on $\mathfrak{g}$ associated with the contact 1-form η (see (4.1), (4.2), (4.5) and (4.6)). Thus, using Theorem 4.8 and the fact that $X_0 \in \mathcal{Z}(\mathfrak{g})$, we can define a Lie bracket $[\,,\,]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ in such a way that the pair $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra.

On the other hand, from Corollary 4.11 and since r is a solution of the classical Yang-Baxter equation on $\mathfrak{h}$, it follows that there exists a Lie bracket $[\,,\,]_{\mathfrak{h}^*}$ on $\mathfrak{h}^*$ in such a way that the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra. In fact, the Lie algebras $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ and $(\mathfrak{h}^*, [\,,\,]_{\mathfrak{h}^*})$ are isomorphic and, using (4.17), we get that $[(\mu,\lambda),(\nu,\gamma)]_{\mathfrak{g}^*} = ([\mu,\nu]_{\mathfrak{h}^*},0)$, for $(\mu,\lambda),(\nu,\gamma) \in \mathfrak{h}^*\oplus\mathbb{R} \cong \mathfrak{g}^*$. Consequently, $\mathfrak{g}^*$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h}\oplus\mathbb{R}$.

We illustrate the preceding construction with a simple example.

Let $(\mathfrak{h},[\,,\,]_{\mathfrak{h}})$ be the abelian Lie algebra of dimension $2n$ and Ω the usual symplectic 2-form. Then, $\mathfrak{h}\oplus\mathbb{R}$ endowed with the Lie bracket given by (4.23) is just the Lie algebra $\mathfrak{h}(1,n)$ of the generalized Heisenberg group $\mathbb{H}(1,n)$ (see [41]) and the 1-form η is just the usual algebraic contact 1-form on $\mathfrak{h}(1,n)$. In this case, the Lie algebra $\mathfrak{h}(1,n)^*$ is abelian.

Remark 4.13 A complete description of symplectic Lie algebras of dimension 4 was obtained in [92] (for a detailed study of symplectic Lie algebras, see also [21, 76]). Thus, one can determine all contact Lie algebras of dimension 5 with center of dimension 1 and from there, using Theorem 4.8, obtain different examples of Jacobi bialgebras.

Now, we will give two examples of Jacobi bialgebras $((\mathfrak{g},\phi_0),\,(\mathfrak{g}^*,X_0))$ associated with an algebraic contact structure on $\mathfrak{g}$ but in both cases $\phi_0\neq 0$. In the first example, $X_0\in\mathcal{Z}(\mathfrak{g})$. However, $X_0\notin\mathcal{Z}(\mathfrak{g})$ in the second one.

1.- Let $(\mathfrak{h},[\,,\,]_{\mathfrak{h}})$ be the nonabelian solvable Lie algebra of dimension 2. We can find a basis $\{e_1,e_2\}$ of $\mathfrak{h}$ such that $[e_1,e_2]_{\mathfrak{h}} = e_1$. If we consider on $\mathfrak{g}=\mathfrak{h}\oplus\mathbb{R}$ the Lie bracket given by (4.23), it is easy to prove that $\phi_0=-e^2$ is a 1-cocycle of $\mathfrak{g}$, $\{e^1,e^2\}$ being the dual basis of $\{e_1,e_2\}$. We also have that $\eta=(0,1)\in\mathfrak{h}^*\oplus\mathbb{R}\cong\mathfrak{g}^*$ is an algebraic contact 1-form on $\mathfrak{g}$ and that (r,X_0') is the corresponding Jacobi structure, where $r=e_2\wedge e_1$ and $X_0'=(0,1)\in\mathfrak{h}\oplus\mathbb{R}=\mathfrak{g}$. On the other hand, using (4.23), we deduce that $i_{\phi_0}r-X_0'$ is $ad^{\mathfrak{g}}_{(\phi_0,0)}$-invariant. Thus, from Theorem 4.8, $((\mathfrak{g},\phi_0),(\mathfrak{g}^*,-X_0'))$ is a Jacobi bialgebra. Note that the Lie algebra $(\mathfrak{g},[\,,\,]_{\mathfrak{g}})$ is isomorphic to the direct product $\mathfrak{h}\oplus\mathbb{R}$ and that $\mathfrak{g}^*$ is the abelian Lie algebra of dimension 3 (see (4.17)).

2.- Let $(\mathfrak{g},[\,,\,]_{\mathfrak{g}})$ be the solvable Lie algebra of dimension 3 with basis $\{e_1,e_2,$ $e_3\}$ such that

$$[e_1,e_2]_{\mathfrak{g}}=0,\quad [e_1,e_3]_{\mathfrak{g}}=e_1,\quad [e_3,e_2]_{\mathfrak{g}}=e_2.$$

Take $r = e_3 \wedge (e_1 - e_2)$ and $X_0' = e_1 + e_2$. It is easy to prove that (r, X_0') is an algebraic Jacobi structure on $\mathfrak{g}$ which is associated with an algebraic contact structure. Moreover, if $\{e^1, e^2, e^3\}$ is the dual basis of $\mathfrak{g}^*$ then $\phi_0 = e^3$ is a 1-cocycle of $\mathfrak{g}$ and $i_{\phi_0} r - X_0'$ is $ad^{\mathfrak{g}}_{(\phi_0,0)}$-invariant. Therefore, from Theorem 4.8, we deduce that $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is a Jacobi bialgebra. The Lie bracket on $\mathfrak{g}^*$ is characterized by

$$[e^1, e^2]_{\mathfrak{g}^*} = 0, \quad [e^1, e^3]_{\mathfrak{g}^*} = e^3, \quad [e^2, e^3]_{\mathfrak{g}^*} = -e^3.$$

4.3.2 Jacobi bialgebras from locally conformal symplectic Lie algebras

Suppose that $(r_{\mathfrak{h}}, X_0')$ is an algebraic contact structure on a Lie algebra $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$. If we consider on the direct product of Lie algebras $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ the 2-vector

$$r = r_{\mathfrak{h}} + e_0 \wedge X_0', \tag{4.24}$$

where $e_0 = (0,1) \in \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}$, then (r, X_0') is an algebraic l.c.s. structure and, using Remark 4.12, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is a Jacobi bialgebra, with $\phi_0 = (0,1) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. In addition, the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are isomorphic (see Remark 4.12).

Remark 4.14 If H is a connected Lie group with Lie algebra $\mathfrak{h}$ then the pair $(\overleftarrow{r}, \overleftarrow{X'}_0)$ defines, on the direct product $G = H \times \mathbb{R}$, a left invariant l.c.s. structure of the first kind in the sense of Vaisman [109].

In the case when $\mathcal{Z}(\mathfrak{h}) = < X_0' >$ we have that the pair $((\mathfrak{h}, 0), (\mathfrak{h}^*, -X_0'))$ is a Jacobi bialgebra (see Section 4.3.1). Moreover, from (4.17) and (4.24), we deduce that the Lie bracket $[\,,\,]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ can be described, in terms of the Lie bracket $[\,,\,]_{\mathfrak{h}^*}$ of $\mathfrak{h}^*$, as follows

$$[(\mu, \lambda), (\nu, \mu)]_{\mathfrak{g}^*} = ([\mu, \nu]_{\mathfrak{h}^*}, -r_{\mathfrak{h}}(\mu, \nu)),$$

for $(\mu, \lambda), (\nu, \mu) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. Thus, since $r_{\mathfrak{h}}$ is a 2-cocycle of the Lie algebra $(\mathfrak{h}^*, [\,,\,]_{\mathfrak{h}^*})$ (see (4.21)), it follows that $\mathfrak{g}^*$ is the central extension of $\mathfrak{h}^*$ by $\mathbb{R}$ via the 2-cocycle $r_{\mathfrak{h}}$.

On the other hand, in [25], Diatta proved that if $(\mathfrak{h}', [\,,\,]_{\mathfrak{h}'})$ is an exact symplectic Lie algebra then one can define on the direct product $\mathfrak{h} = \mathfrak{h}' \oplus \mathbb{R}$ a Lie bracket in such a way that $\mathfrak{h}$ is a contact Lie algebra, with trivial center, and $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$. Using this construction we can also obtain different examples of Jacobi bialgebras. Next, we will show an explicit example.

Let $\mathfrak{sl}(2,\mathbb{R})$ be the Lie algebra of the special linear group $SL(2,\mathbb{R})$. Then, there exists a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{sl}(2,\mathbb{R})$ such that

$$[e_1, e_2]_{\mathfrak{sl}(2,\mathbb{R})} = 2e_2, \quad [e_3, e_1]_{\mathfrak{sl}(2,\mathbb{R})} = 2e_3, \quad [e_2, e_3]_{\mathfrak{sl}(2,\mathbb{R})} = e_1.$$

It is clear that $\mathfrak{sl}(2,\mathbb{R})$ admits exact symplectic Lie subalgebras and, therefore, we can apply Diatta's method in order to obtain algebraic contact structures on $\mathfrak{sl}(2,\mathbb{R})$. In fact, if λ^1, λ^2 and λ^3 are real numbers satisfying the relation $(\lambda^1)^2 + 4\lambda^2\lambda^3 \neq 0$ then the pair $(r_{\mathfrak{sl}(2,\mathbb{R})}, X_0')$ given by

$$r_{\mathfrak{sl}(2,\mathbb{R})} = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_1 \wedge e_2 + \lambda^3 e_3 \wedge e_1, \quad X_0' = -(\lambda^1 e_1 + 2\lambda^2 e_2 + 2\lambda^3 e_3),$$

defines an algebraic Jacobi structure on $\mathfrak{sl}(2,\mathbb{R})$ which is associated with an algebraic contact structure. Consequently, since $\mathfrak{gl}(2,\mathbb{R})$ (the Lie algebra of the general linear group $GL(2,\mathbb{R})$) is isomorphic to the direct product $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$, we conclude that the pair $((\mathfrak{gl}(2,\mathbb{R}), \phi_0), (\mathfrak{gl}(2,\mathbb{R})^*, -X_0'))$ is a Jacobi bialgebra, where $\phi_0 = (0,1) \in \mathfrak{sl}(2,\mathbb{R})^* \oplus \mathbb{R} \cong \mathfrak{gl}(2,\mathbb{R})^*$.

Finally, we remark that there exist examples of contact Lie algebras with trivial center which do not admit symplectic Lie subalgebras. An interesting case is $\mathfrak{su}(2)$, the Lie algebra of the special unitary group $SU(2)$. We can consider a basis $\{e_1, e_2, e_3\}$ of $\mathfrak{su}(2)$ such that

$$[e_1, e_2]_{\mathfrak{su}(2)} = e_3, \quad [e_3, e_1]_{\mathfrak{su}(2)} = e_2, \quad [e_2, e_3]_{\mathfrak{su}(2)} = e_1.$$

Then, if λ^1, λ^2 and λ^3 are real numbers, $(\lambda^1, \lambda^2, \lambda^3) \neq (0,0,0)$, we have that the pair $(r_{\mathfrak{su}(2)}, X_0')$ given by

$$r_{\mathfrak{su}(2)} = \lambda^1 e_2 \wedge e_3 + \lambda^2 e_3 \wedge e_1 + \lambda^3 e_1 \wedge e_2, \quad X_0' = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

defines an algebraic Jacobi structure on $\mathfrak{su}(2)$ which is associated with an algebraic contact structure. Thus, since $\mathfrak{u}(2)$ (the Lie algebra of the unitary group $U(2)$) is isomorphic to the direct product $\mathfrak{su}(2) \oplus \mathbb{R}$, we deduce that the pair $((\mathfrak{u}(2), \phi_0), (\mathfrak{u}(2)^*, -X_0'))$ is a Jacobi bialgebra, where $\phi_0 = (0, 1) \in \mathfrak{su}(2)^* \oplus \mathbb{R} \cong \mathfrak{u}(2)^*$.

We will treat again this example in Section 4.4.

4.3.3 Other examples of Jacobi bialgebras

All the examples of Jacobi bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ considered in Sections 4.3.1 and 4.3.2 have been obtained from an algebraic Jacobi structure $(r, -X_0)$ on $\mathfrak{g}$. However, the hypotheses of Theorem 4.8 do not necessarily imply that the pair $(r, -X_0)$ is an algebraic Jacobi structure on $\mathfrak{g}$, as it is shown in the following simple example.

Let $\mathfrak{h}$ be the abelian Lie algebra of dimension 3. Take $\{e_1, e_2, e_3\}$ a basis of $\mathfrak{h}$ and let $\{e^1, e^2, e^3\}$ be the dual basis of $\mathfrak{h}^*$. Denote by Ψ the endomorphism of $\mathfrak{h}$ given by $\Psi = \frac{1}{2} e_1 \otimes e^1 + \frac{1}{2} e_2 \otimes e^2 + e_3 \otimes e^3$. Ψ is a 1-cocycle with respect to the adjoint representation of $\mathfrak{h}$. Thus, we can consider the representation of $\mathbb{R}$ on $\mathfrak{h}$ given by $\mathbb{R} \times \mathfrak{h} \to \mathfrak{h}$, $(\lambda, X) \mapsto \lambda \Psi(X)$, and the corresponding semi-direct product $\mathfrak{g} = \mathfrak{h} \times_\Psi \mathbb{R}$. We can choose a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathfrak{g}$ such that

$$[e_4, e_1]_{\mathfrak{g}} = \frac{1}{2} e_1, \quad [e_4, e_2]_{\mathfrak{g}} = \frac{1}{2} e_2, \quad [e_4, e_3]_{\mathfrak{g}} = e_3,$$

and the other brackets are zero. Suppose that $\{e^1, e^2, e^3, e^4\}$ is the dual basis of $\mathfrak{g}^*$. If $r \in \wedge^2 \mathfrak{g}$, $X_0 \in \mathfrak{g}$ and $\phi_0 \in \mathfrak{g}^*$ are defined by

$$r = e_1 \wedge e_2 - 2 e_3 \wedge e_4, \quad X_0 = -e_3, \quad \phi_0 = e^4,$$

then r, X_0 and ϕ_0 satisfy the hypotheses of Theorem 4.8. However, $[r, r]_{\mathfrak{g}} + 2X_0 \wedge r = 2 e_1 \wedge e_2 \wedge e_3 \neq 0$ and $i_{\phi_0} r + X_0 = e_3 \neq 0$. Moreover, a direct computation shows that,

$$[e^3, e^4]_{\mathfrak{g}^*} = -e^4, \quad [e^i, e^j]_{\mathfrak{g}^*} = 0,$$

for $1 \leq i < j \leq 4$, $(i, j) \neq (3, 4)$.

Finally, we will exhibit an example of a Jacobi bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ such that $\phi_0 \neq 0$ and $d_*^{X_0}$ is not a 1-coboundary with respect to the representation $ad_{(\phi_0,1)}: \mathfrak{g} \times \wedge^2\mathfrak{g} \to \wedge^2\mathfrak{g}$. On the other hand, all the examples of Jacobi bialgebras that we have given in Section 4.3 are such that $d_*^{X_0}$ is a 1-coboundary.

Let $\mathfrak{g}$ be the Lie algebra of dimension 4 with basis $\{e_1, e_2, e_3, e_4\}$ satisfying

$$[e_4, e_1]_{\mathfrak{g}} = e_1, \quad [e_4, e_2]_{\mathfrak{g}} = e_2, \quad [e_4, e_3]_{\mathfrak{g}} = e_3$$

and the other brackets being zero. If $\{e^1, e^2, e^3, e^4\}$ is the dual basis of $\mathfrak{g}^*$, we consider on $\mathfrak{g}^*$ the Lie bracket $[\, ,\,]_{\mathfrak{g}^*}$ characterized by

$$[e^1, e^2]_{\mathfrak{g}^*} = e^3, \quad [e^1, e^4]_{\mathfrak{g}^*} = e^4, \quad [e^i, e^j]_{\mathfrak{g}^*} = 0,$$

for $1 \leq i < j \leq 4$, $(i, j) \neq (1, 2), (1, 4)$. Then, the pair $((\mathfrak{g}, e^4), (\mathfrak{g}^*, e_1))$ is a Jacobi bialgebra. Moreover, it is easy to prove that there does not exist $r \in \wedge^2\mathfrak{g}$ such that $d_*^{X_0}X = ad_{(\phi_0,1)}(X)(r)$, for all $X \in \mathfrak{g}$.

4.4 Compact Jacobi bialgebras

Several authors have devoted special attention to the study of compact Lie bialgebras and an important result in this direction is the following one [81] (see also [86]): every connected compact semisimple Lie group has a nontrivial Poisson Lie group structure.

In this Section, we will describe the structure of a Jacobi bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, $\mathfrak{g}$ being a compact Lie algebra (that is, $\mathfrak{g}$ is the Lie algebra of a compact connected Lie group).

If $\phi_0 = 0$ and $X_0 = 0$, the pair $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. Thus, we will suppose that $\phi_0 \neq 0$ or $X_0 \neq 0$. Note that if $\phi_0 = 0$ then $X_0 \in \mathcal{Z}(\mathfrak{g})$ (see (4.11)). On the other hand, if $\phi_0 \neq 0$ then we can consider an ad-invariant scalar product $\langle\, ,\, \rangle: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ and the vector $\bar{Y}_0 \in \mathfrak{g}$ characterized by the relation

$\phi_0(X) = \langle X, \bar{Y}_0\rangle$, for $X \in \mathfrak{g}$. It is clear that $\bar{Y}_0 \neq 0$ and, moreover, using that ϕ_0 is a 1-cocycle and the fact that $\langle\, , \,\rangle$ is an *ad*-invariant scalar product, we obtain that $\bar{Y}_0 \in \mathcal{Z}(\mathfrak{g})$ (we remark that $\phi_0(Y_0) = 1$ with $Y_0 = \frac{\bar{Y}_0}{\phi_0(\bar{Y}_0)} \in \mathcal{Z}(\mathfrak{g})$). Therefore, if $\phi_0 \neq 0$ or $X_0 \neq 0$, we have that $dim\, \mathcal{Z}(\mathfrak{g}) \geq 1$. This implies that a compact connected Lie group G with Lie algebra $\mathfrak{g}$ cannot be semisimple.

Next, we will distinguish two cases:

a) The case $\phi_0 \neq 0$

Let $\mathfrak{g}$ be a compact Lie algebra and $\phi_0 \in \mathfrak{g}^*$ a 1-cocycle, $\phi_0 \neq 0$. If $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$ and $(r, i_{\phi_0}(r))$ is an algebraic l.c.s. structure on $\mathfrak{h}$ then, from Corollary 4.11, we deduce that the pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -i_{\phi_0}(r)))$ is a Jacobi bialgebra, where the Lie bracket on $\mathfrak{g}^*$ is given by (4.17).

Using the above construction, we can obtain some examples of Jacobi bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra.

Examples 4.15 1.- *Compact Jacobi bialgebras of the first kind*
Let $\mathfrak{g}$ be a compact Lie algebra and $\mathfrak{h}$ an abelian Lie subalgebra of even dimension. Furthermore, assume that $r \in \wedge^2\mathfrak{h}$ is a nondegenerate 2-vector on $\mathfrak{h}$ (that is, r comes from an algebraic symplectic structure on $\mathfrak{h}$) and that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ such that $\phi_0 \neq 0$ and $\phi_0 \in \mathfrak{h}^\circ$, $\mathfrak{h}^\circ$ being the annihilator of $\mathfrak{h}$. Then, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, 0))$ is a Jacobi bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, 0))$ is said to be a compact Jacobi bialgebra of the first kind.

2.- *Compact Jacobi bialgebras of the second kind*
Let $(\mathfrak{g}, [\, , \,]_{\mathfrak{g}})$ be a compact real Lie algebra. Suppose that $e_1, e_2 \in \mathfrak{g}$ are linearly independent and that $[e_1, e_2]_{\mathfrak{g}} = 0$. We consider the 2-vector r and the vector X_0 on $\mathfrak{g}$ defined by $r = \lambda e_1 \wedge e_2$ and $X_0' = \lambda^1 e_1 + \lambda^2 e_2$, with $\lambda \in \mathbb{R} - \{0\}$ and $(\lambda^1, \lambda^2) \in \mathbb{R}^2 - \{(0,0)\}$. It is clear that (r, X_0') is an algebraic Jacobi structure on $\mathfrak{g}$ which comes from an algebraic l.c.s. structure on the Lie subalgebra $\mathfrak{h} =< e_1, e_2 >$. Therefore, if $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle of $\mathfrak{g}$ such that $i_{\phi_0}(r) = X_0'$ (that is, $\phi_0(e_1) = \frac{\lambda^2}{\lambda}$ and $\phi_0(e_2) = -\frac{\lambda^1}{\lambda}$) then $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is a Jacobi bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is said to be a compact Jacobi bialgebra of the second kind.

3.- *Compact Jacobi bialgebras of the third kind*
Let $(\mathfrak{g}, [\,,\,]_{\mathfrak{g}})$ be a nonabelian compact real Lie algebra. By the root space decomposition theorem, we know that there exist $e_1, e_2, e_3 \in \mathfrak{g}$ satisfying

$$[e_1, e_2]_{\mathfrak{g}} = e_3, \quad [e_3, e_1]_{\mathfrak{g}} = e_2, \quad [e_2, e_3]_{\mathfrak{g}} = e_1. \tag{4.25}$$

Now, suppose that $\phi_0 \in \mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ and that e_4 is a vector of $\mathfrak{g}$ such that $\phi_0(e_4) = 1$, and $[e_4, e_i]_{\mathfrak{g}} = 0$, for $i = 1, 2, 3$ (note that if $\mathcal{Z}(\mathfrak{g}) \neq \{0\}$, then the existence of ϕ_0 and e_4 is guaranteed). Then, we consider the 2-vector r and the vector X_0' on $\mathfrak{g}$ defined by

$$r = \lambda^1(e_2 \wedge e_3 + e_1 \wedge e_4) + \lambda^2(e_3 \wedge e_1 + e_2 \wedge e_4) + \lambda^3(e_1 \wedge e_2 + e_3 \wedge e_4),$$

$$X_0' = -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3),$$

with $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0, 0, 0)\}$. A direct computation proves that (r, X_0') is an algebraic l.c.s. structure on the Lie subalgebra $\mathfrak{h} =< e_1, e_2, e_3,\ e_4 >$ (see Section 4.3.2). Moreover, $i_{\phi_0}(r) = X_0'$. Thus, $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is a Jacobi bialgebra. The pair $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, -X_0'))$ is said to be a compact Jacobi bialgebra of the third kind.

Next, we will show that Examples 4.15 1, 2 and 3 are the only examples of Jacobi bialgebras $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra.

Theorem 4.16 *Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a Jacobi bialgebra. Suppose that $\phi_0(Y_0) = 1$, with $Y_0 \in \mathcal{Z}(\mathfrak{g})$. Then, there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a 2-vector $r \in \wedge^2\mathfrak{h} \subseteq \wedge^2\mathfrak{g}$ such that $X_0 \in \mathfrak{h}$ and:*

i) The pair $(r, i_{\phi_0}(r))$ defines an algebraic Jacobi structure on $\mathfrak{g}$ which is associated with an algebraic l.c.s. structure on $\mathfrak{h}$. Moreover, $i_{\phi_0}(r) = -X_0$.

ii) The Lie bracket $[\,,\,]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by (4.17).

Proof: Denote by r the 2-vector on $\mathfrak{g}$ given by

$$r = d_*^{X_0} Y_0. \tag{4.26}$$

Using (4.10), (4.11), (4.26) and the fact that $Y_0 \in \mathcal{Z}(\mathfrak{g})$, we have that

$$i_{\phi_0}(r) = -X_0. \tag{4.27}$$

From (4.9), (4.26) and since $Y_0 \in \mathcal{Z}(\mathfrak{g})$, it follows that

$$0 = d_*^{X_0}[X, Y_0]_{\mathfrak{g}} = [X, r]_{\mathfrak{g}} - \phi_0(X)r + d_*^{X_0}X, \tag{4.28}$$

for all $X \in \mathfrak{g}$. Therefore, using (4.10), (4.28) and the fact that X_0 is a 1-cocycle on $(\mathfrak{g}^*, [\,,\,]_{\mathfrak{g}^*})$, we deduce that

$$[X_0, r]_{\mathfrak{g}} = 0. \tag{4.29}$$

On the other hand, using again (4.28) and the properties of the algebraic Schouten bracket $[\,,\,]_{\mathfrak{g}}$, we conclude that $[r', r]_{\mathfrak{g}} = -d_* r' - 2X_0 \wedge r' + r \wedge i_{\phi_0}(r')$, for $r' \in \wedge^2\mathfrak{g}$. Consequently (see (4.26) and (4.27)),

$$[r, r]_{\mathfrak{g}} - 2i_{\phi_0}(r) \wedge r = -(d_* r + r \wedge X_0) = -d_*^{X_0} r = 0. \tag{4.30}$$

Thus, the pair $(r, i_{\phi_0}(r))$ is an algebraic Jacobi structure on $\mathfrak{g}$ and the rank of $(r, i_{\phi_0}(r))$ is even (see (4.27), (4.29) and (4.30)). Therefore, using Proposition 4.4, it follows that there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $r \in \wedge^2\mathfrak{h}$, $X_0 = -i_{\phi_0}(r) \in \mathfrak{h}$ and the pair $(r, i_{\phi_0}(r))$ is associated with an algebraic l.c.s. structure on $\mathfrak{h}$.

Finally, from (4.22) and (4.28), we deduce that the Lie bracket on $\mathfrak{g}^*$ is given by (4.17). *QED*

Now, we will describe the algebraic l.c.s. structures on a compact Lie algebra.

Theorem 4.17 *Let $\mathfrak{h}$ be a compact Lie algebra of dimension $2k \geq 2$. Suppose that (r, X_0') is an algebraic Jacobi structure on $\mathfrak{h}$ which is associated with an algebraic l.c.s. structure.*

i) If $X_0' = 0$ then $\mathfrak{h}$ is the abelian Lie algebra and r is a nondegenerate 2-vector on $\mathfrak{h}$.

ii) If $X_0' \neq 0$ and $k = 1$ then $\mathfrak{h}$ is the abelian Lie algebra and r is an arbitrary 2-vector on $\mathfrak{h}$, $r \neq 0$.

iii) If $X_0' \neq 0$ and $k \geq 2$ then $k = 2$, $\mathfrak{h}$ is isomorphic to $\mathfrak{u}(2)$ and

$$\begin{aligned} r &= \lambda^1(e_2 \wedge e_3 + e_1 \wedge e_4) + \lambda^2(e_3 \wedge e_1 + e_2 \wedge e_4) \\ &\quad + \lambda^3(e_1 \wedge e_2 + e_3 \wedge e_4), \\ X_0' &= -(\lambda^1 e_1 + \lambda^2 e_2 + \lambda^3 e_3), \end{aligned}$$

where $(\lambda^1, \lambda^2, \lambda^3) \in \mathbb{R}^3 - \{(0,0,0)\}$ and $\{e_1, e_2, e_3, e_4\}$ is a basis of $\mathfrak{h}$ such that $e_4 \in \mathcal{Z}(\mathfrak{h})$ and

$$[e_1, e_2]_{\mathfrak{h}} = e_3, \quad [e_3, e_1]_{\mathfrak{h}} = e_2, \quad [e_2, e_3]_{\mathfrak{h}} = e_1. \tag{4.31}$$

Proof: Denote by (Ω, ω) the algebraic l.c.s. structure on $\mathfrak{h}$ associated with the pair (r, X_0').

i) If $X_0' = 0$, we obtain that $\omega = 0$ and Ω is an algebraic symplectic structure on $\mathfrak{h}$ (see (4.4)). Thus, since $\mathfrak{h}$ is a compact Lie algebra, *i)* follows using the results in [12] (see also [76]).

ii) It is trivial.

iii) Suppose that $X_0' \neq 0$ and that $k \geq 2$. Then, $\omega \neq 0$. Moreover, we can consider an *ad*-invariant scalar product $\langle\, ,\, \rangle \colon \mathfrak{h} \times \mathfrak{h} \to \mathbb{R}$ and the vector $\bar{Y}_0$ of $\mathfrak{h}$ characterized by the relation

$$\omega(X) = \langle X, \bar{Y}_0 \rangle, \text{ for } X \in \mathfrak{h}. \tag{4.32}$$

Using (4.32) and the fact that ω is a 1-cocycle, we deduce that $\bar{Y}_0 \in \mathcal{Z}(\mathfrak{h})$. Consequently,

$$\omega(Y_0) = 1, \tag{4.33}$$

with $Y_0 = \frac{\bar{Y}_0}{\omega(\bar{Y}_0)} \in \mathcal{Z}(\mathfrak{h})$.

On the other hand, if $\mathfrak{h}' \subseteq \mathfrak{h}$ is the annihilator of the subspace generated by ω, it is clear that $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$. In fact, using (4.33) and since $Y_0 \in \mathcal{Z}(\mathfrak{h})$ and ω is a 1-cocycle, it follows that $\mathfrak{h}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h}' \oplus \mathbb{R}$. In addition, we will show that $\mathfrak{h}'$

admits an algebraic contact structure. For this purpose, we define the 1-form $\bar{\eta}$ on $\mathfrak{h}$ given by

$$\bar{\eta} = -i_{Y_0}\Omega. \tag{4.34}$$

Using the equality $\omega = i_{X'_0}\Omega$, we have that

$$\bar{\eta}(X'_0) = 1. \tag{4.35}$$

Moreover, from (4.4), (4.33), (4.34) and since $Y_0 \in \mathcal{Z}(\mathfrak{h})$, we deduce that

$$0 = \mathcal{L}_{Y_0}\Omega = i_{Y_0}(d\Omega) + d(i_{Y_0}\Omega) = \Omega + \omega \wedge \bar{\eta} - d\bar{\eta}. \tag{4.36}$$

In particular (see (4.33), (4.34) and (4.35))

$$i_{X'_0}(d\bar{\eta}) = i_{Y_0}(d\bar{\eta}) = 0. \tag{4.37}$$

Thus, the condition $\Omega^k = \Omega \wedge \overset{(k}{\ldots} \wedge \Omega \neq 0$ implies that $\omega \wedge \bar{\eta} \wedge (d\bar{\eta})^{k-1} \neq 0$. Therefore, the restriction η of $\bar{\eta}$ to $\mathfrak{h}'$ is an algebraic contact 1-form on $\mathfrak{h}'$. Furthermore, if $(\hat{r}, \hat{X}_0)$ is the algebraic Jacobi structure on $\mathfrak{h}'$ associated with the contact 1-form η then, from relations (4.33)-(4.37) and the results in Section 4.1, we obtain that $\hat{r} = r + Y_0 \wedge X'_0$ and $\hat{X}_0 = X'_0$. Consequently, taking $e_4 = -Y_0$ and using Proposition 4.5, we prove *iii)*. QED

Now, suppose that $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. Under these conditions we showed, at the beginning of this Section, that there exists $Y_0 \in \mathcal{Z}(\mathfrak{g})$ satisfying that $\phi_0(Y_0) = 1$. Then, using Theorems 4.16 and 4.17, we deduce the following result.

Theorem 4.18 *Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a Jacobi bialgebra, with $\phi_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. If $X_0 = 0$ (respectively, $X_0 \neq 0$) then it is of the first kind (respectively, the second or third kind).*

b) The case $\phi_0 = 0$

We will describe the structure of a Jacobi bialgebra $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$, $\mathfrak{g}$ being a compact Lie algebra and $X_0 \neq 0$. First, we will examine a suitable example.

Let $(\mathfrak{h}, \mathfrak{h}^*)$ be a Lie bialgebra and Ψ be an endomorphism of $\mathfrak{h}$, $\Psi \colon \mathfrak{h} \to \mathfrak{h}$. Assume that Ψ is a 1-cocycle of $\mathfrak{h}$ with respect to the adjoint representation

$ad^{\mathfrak{h}}: \mathfrak{h} \times \mathfrak{h} \to \mathfrak{h}$ and that $\Psi^* - Id$ is a 1-cocycle of $\mathfrak{h}^*$ with respect to the adjoint representation $ad^{\mathfrak{h}^*}: \mathfrak{h}^* \times \mathfrak{h}^* \to \mathfrak{h}^*$. Here, $\Psi^*: \mathfrak{h}^* \to \mathfrak{h}^*$ is the adjoint linear map of $\Psi: \mathfrak{h} \to \mathfrak{h}$. Denote by $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ the direct product of the Lie algebras $\mathfrak{h}$ and $\mathbb{R}$ and consider on $\mathfrak{g}^* \cong \mathfrak{h}^* \oplus \mathbb{R}$ the Lie bracket $[\, ,\,]_{\mathfrak{g}^*}$ defined by

$$[(\mu, \lambda), (\nu, \gamma)]_{\mathfrak{g}^*} = ([\mu, \nu]_{\mathfrak{h}^*} - \lambda(\Psi^* - Id)(\nu) + \gamma(\Psi^* - Id)(\mu), 0), \qquad (4.38)$$

for $(\mu, \lambda), (\nu, \gamma) \in \mathfrak{h}^* \oplus \mathbb{R} \cong \mathfrak{g}^*$. Then, we have that $X_0 = (0, 1) \in \mathfrak{h} \oplus \mathbb{R} = \mathfrak{g}$ is a 1-cocycle of $(\mathfrak{g}^*, [\, ,\,]_{\mathfrak{g}^*})$ and that $X_0 \in \mathcal{Z}(\mathfrak{g})$. Thus, using (4.38), that $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra and the fact that Ψ is a 1-cocycle, we deduce that the pair $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$ is a Jacobi bialgebra. Moreover, it is clear that if $\mathfrak{h}$ is a compact Lie algebra then $\mathfrak{g}$ is also compact.

Next, suppose that $\mathfrak{h}$ is compact and semisimple. Then, as we know (see [80]), if $\Phi : \mathfrak{h} \times V \to V$ is a representation of $\mathfrak{h}$ on a vector space V, every 1-cocycle $\epsilon : \mathfrak{h} \to V$ is a 1-coboundary, that is, $\epsilon(X) = \Phi(X, v_0)$, for some $v_0 \in V$. Therefore, if $d_{\mathfrak{h}^*}$ is the Chevalley-Eilenberg differential of $\mathfrak{h}^*$, it follows that there exist $r \in \wedge^2\mathfrak{h}$ and $Z \in \mathfrak{h}$ such that

$$d_{\mathfrak{h}^*}X = -[X, r]_{\mathfrak{h}}, \quad \Psi(X) = [X, Z]_{\mathfrak{h}}, \quad \Psi^*(\alpha) = coad^{\mathfrak{h}}_Z\alpha = \mathcal{L}_Z\alpha, \qquad (4.39)$$

for $X \in \mathfrak{h}$ and $\alpha \in \mathfrak{h}^*$, where $coad^{\mathfrak{h}}: \mathfrak{h} \times \mathfrak{h}^* \to \mathfrak{h}^*$ is the coadjoint representation. Using (4.39) and the fact that $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$, we deduce that

$$([[X, Z]_{\mathfrak{h}}, r]_{\mathfrak{h}} + [Z, [X, r]_{\mathfrak{h}}]_{\mathfrak{h}})(\mu, \nu) = (d_{\mathfrak{h}^*}X)(\mu, \nu) = -[X, r]_{\mathfrak{h}}(\mu, \nu),$$

for $\mu, \nu \in \mathfrak{h}^*$. Thus, the equality $[X, [Z, r]_{\mathfrak{h}}]_{\mathfrak{h}} = [[X, Z]_{\mathfrak{h}}, r]^{\mathfrak{h}} + [Z, [X, r]_{\mathfrak{h}}]_{\mathfrak{h}}$ implies that

$$[X, [Z, r]_{\mathfrak{h}}]_{\mathfrak{h}} = -[X, r]_{\mathfrak{h}}, \text{ for all } X \in \mathfrak{h}. \qquad (4.40)$$

The compact character of $\mathfrak{h}$ allows us to choose an $ad^{\mathfrak{h}}$-invariant scalar product $\langle\, ,\, \rangle$ on $\mathfrak{h}$. We will also denote by $\langle\, ,\, \rangle$ the natural extension of $\langle\, ,\, \rangle$ to $\wedge^2\mathfrak{h}$. This extension is a scalar product on $\wedge^2\mathfrak{h}$ and, in addition, it is easy to prove that $\langle [X,s]_{\mathfrak{h}}, t\rangle = -\langle s, [X,t]_{\mathfrak{h}}\rangle$, for $X \in \mathfrak{h}$ and $s,t \in \wedge^2\mathfrak{h}$. Thus (see (4.40)),

$$\langle [Z,r]_{\mathfrak{h}}, [Z,r]_{\mathfrak{h}}\rangle = -\langle r, [Z,[Z,r]_{\mathfrak{h}}]_{\mathfrak{h}}\rangle = \langle r, [Z,r]_{\mathfrak{h}}\rangle = 0,$$

i.e.,

$$[Z,r]_{\mathfrak{h}} = 0. \tag{4.41}$$

Then, from (4.39), (4.40) and (4.41), we conclude that the Lie bracket $[\,,\,]_{\mathfrak{h}^*}$ is trivial.

Remark 4.19 If $\mathfrak{h}$ is not semisimple then the Lie bracket $[\,,\,]_{\mathfrak{h}^*}$ is not, in general, trivial. In fact, suppose that $\mathcal{Z}(\mathfrak{h}) \neq \{0\}$. We know that $\mathfrak{h}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h})$, where $\mathfrak{h}'$ is a compact semisimple Lie subalgebra of $\mathfrak{h}$. Therefore, if $\Psi\colon \mathfrak{h} \cong \mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h}) \to \mathfrak{h} \cong \mathfrak{h}' \oplus \mathcal{Z}(\mathfrak{h})$ is the projection on the subspace $\mathcal{Z}(\mathfrak{h})$, it follows that Ψ is an adjoint 1-cocycle of $\mathfrak{h}$. Furthermore, if on $(\mathfrak{h}')^*$ we consider the trivial Lie bracket and on $\mathcal{Z}(\mathfrak{h})^*$ an arbitrary (nontrivial) Lie bracket then the direct product $(\mathfrak{h}')^* \oplus \mathcal{Z}(\mathfrak{h})^* \cong \mathfrak{h}^*$ is a Lie algebra, the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra and the endomorphism $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$.

Now, we prove

Theorem 4.20 *Let $((\mathfrak{g}, 0), (\mathfrak{g}^*, X_0))$ be a Jacobi bialgebra with $X_0 \neq 0$ and $\mathfrak{g}$ a compact Lie algebra. Then:*

i) There exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is isomorphic, as a Lie algebra, to the direct product $\mathfrak{h} \oplus \mathbb{R}$. Moreover, under the above isomorphism, $\mathfrak{h}^*$ is a Lie subalgebra of $\mathfrak{g}^*$, the pair $(\mathfrak{h}, \mathfrak{h}^*)$ is a Lie bialgebra, $X_0 = (0,1) \in \mathfrak{h} \oplus \mathbb{R} \cong \mathfrak{g}$ and the Lie bracket $[\,,\,]_{\mathfrak{g}^*}$ on $\mathfrak{g}^*$ is given by

$$[(\mu,\lambda),(\nu,\gamma)]_{\mathfrak{g}^*} = ([\mu,\nu]_{\mathfrak{h}^*} - \lambda(\Psi^* - Id)(\nu) + \gamma(\Psi^* - Id)(\mu), 0),$$

where $\Psi \in End(\mathfrak{h})$ is an adjoint 1-cocycle of $\mathfrak{h}$ and $\Psi^* - Id$ is an adjoint 1-cocycle of $\mathfrak{h}^*$.

ii) If $dim\, \mathcal{Z}(\mathfrak{g}) = 1$ *then the Lie bracket* $[\,,\,]_{\mathfrak{h}^*}$ *is trivial and there exists* $Z \in \mathfrak{h}$ *such that* $\Psi(X) = [X, Z]_{\mathfrak{h}}$*, for all* $X \in \mathfrak{h}$.

Proof: i) From (4.11) it follows that $X_0 \in \mathcal{Z}(\mathfrak{g})$. We consider an $ad^{\mathfrak{g}}$-invariant scalar product $\langle\,,\,\rangle$ on $\mathfrak{g}$ and the 1-form $\theta_0 \in \mathfrak{g}^*$ defined by $\theta_0(X) = \langle X, X_0\rangle$, for all $X \in \mathfrak{g}$. We have that θ_0 is a 1-cocycle of $\mathfrak{g}$ and we can assume, without the loss of generality, that $\theta_0(X_0) = 1$. Then, using (4.9) and the fact that X_0 is a 1-cocycle of $\mathfrak{g}^*$, we deduce that the Lie subalgebra $\mathfrak{h}$ is the annihilator of the subspace generated by θ_0 and that the endomorphism $\Psi\colon \mathfrak{h} \to \mathfrak{h}$ is given by $\Psi(X) = X - i_{\theta_0}(d_* X)$, where d_* is the Chevalley-Eilenberg differential of $\mathfrak{g}^*$.

ii) If $dim\, \mathcal{Z}(\mathfrak{g}) = 1$ then $\mathfrak{h}$ is compact and semisimple and the result follows.

QED

CHAPTER 5

Jacobi groupoids and Jacobi bialgebroids

In this last Chapter, we introduce Jacobi groupoids as a generalization of Poisson and contact groupoids. Then, it is proved that Jacobi bialgebroids are the infinitesimal invariants of Jacobi groupoids.

5.1 Contact groupoids and 1-jet bundles

In this first Section of Chapter 5, we will discuss contact groupoids, developing some of its properties. Moreover, we will introduce two Lie groupoid structures which will be important in the sequel.

First, we will recall the notion of a contact groupoid

Definition 5.1 *[56] Let $G \rightrightarrows M$ be a Lie groupoid, $\eta \in \Omega^1(G)$ be a contact 1-form on G and $\sigma : G \to \mathbb{R}$ be an arbitrary function. If $\oplus_{TG}$ is the partial multiplication in the tangent Lie groupoid $TG \rightrightarrows TM$, we will say that $(G \rightrightarrows M, \eta, \sigma)$ is a* contact groupoid *if and only if*

$$\eta(gh)(X_g \oplus_{TG} Y_h) = \eta(g)(X_g) + e^{\sigma(g)}\eta(h)(Y_h), \tag{5.1}$$

for $(X_g, Y_h) \in TG^{(2)}$.

Remark 5.2 Actually, the definition of a contact groupoid given in [56] is slightly different to the one given here. The relation between both approaches is the following one. If $(G \rightrightarrows M, \theta, \kappa)$ is a contact groupoid in the sense of [56] then $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid in the sense of Definition 5.1, where $\sigma(g) = \kappa(g^{-1})$ for $g \in G$, and $\eta(g)$ is the inverse of $\theta(g^{-1})$ in the cotangent Lie groupoid $T^*G \rightrightarrows A^*G$.

If $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid then, using the associativity of $\oplus_{TG}$, we deduce that $\sigma : G \to \mathbb{R}$ is a multiplicative function, that is,

$$\sigma(gh) = \sigma(g) + \sigma(h), \tag{5.2}$$

for $(g, h) \in G^{(2)}$. In particular, if $\epsilon : M \to G$ is the inclusion then $\sigma_{|\epsilon(M)} \equiv 0$ and therefore, using (5.1), it follows that

$$\eta(\epsilon(x))(\epsilon_*^x(X_x)) = 0,$$

for $x \in M$ and $X_x \in T_xM$. Thus, if $\iota : G \to G$ is the inversion of G, we obtain that $\iota^*\eta = -e^{-\sigma}\eta$. This implies that G is a contact groupoid in the sense of [23]. Using this fact, we deduce the following result.

Proposition 5.3 *Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid and suppose that $dim\, G = 2n + 1$. Then:*

i) If g and h are composable elements of G, we have that

$$\begin{aligned}(d_0\eta)(gh)(X_g \oplus_{TG} Y_h, X'_g \oplus_{TG} Y'_h)& \\ = (d_0\eta)(g)(X_g, X'_g) &+ e^{\sigma(g)}(d_0\eta)(h)(Y_h, Y'_h) \\ &+ e^{\sigma(g)}(X_g(\sigma)\eta(h)(Y'_h) - X'_g(\sigma)\eta(h)(Y_h)),\end{aligned} \tag{5.3}$$

for $(X_g, Y_h), (X'_g, Y'_h) \in TG^{(2)}$.

ii) $M \cong \epsilon(M)$ is a Legendre submanifold of G, that is, $\epsilon^*\eta = 0$ and $dim\, \epsilon(M) = dim\, M = n$.

iii) *If* (Λ, E) *is the Jacobi structure associated with the contact 1-form* η*, then* E *is a right-invariant vector field on* G *and* $E(\sigma) = 0$*. Moreover, if* $X_0 \in \Gamma(AG)$ *is the section of the Lie algebroid* AG *of* G *satisfying* $E = -\overrightarrow{X_0}$*, we have that*

$$\#_\Lambda(d_0\sigma) = \overrightarrow{X_0} - e^{-\sigma}\overleftarrow{X_0}. \tag{5.4}$$

iv) *If* α^T*,* β^T *and* ϵ^T *(respectively,* $\tilde{\alpha}$*,* $\tilde{\beta}$ *and* $\tilde{\epsilon}$*) are the projections and the inclusion in the Lie groupoid* $TG \rightrightarrows TM$ *(respectively,* $T^*G \rightrightarrows A^*G$*) then,*

$$e^{-\sigma}\#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\alpha} = \epsilon^T \circ \alpha^T \circ \#_\Lambda, \qquad \#_\Lambda \circ \tilde{\epsilon} \circ \tilde{\beta} = \epsilon^T \circ \beta^T \circ \#_\Lambda.$$

Proof: Using the results in [23], we directly deduce *i)*, *ii)* and *iii)*.

Now, we will prove *iv)*. Suppose that $\mu_g \in T_g^*G$. Then, from *ii)* and Remark 1.1, we conclude that

$$\eta(\epsilon(\alpha(g)))(e^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\mu_g)))) = \eta(\epsilon(\alpha(g)))(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\mu_g)))) = 0.$$

Furthermore, if $X_{\alpha(g)} \in A_{\alpha(g)}G$, it follows that (see (1.55))

$$\begin{aligned} &\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\mu_g))) = \iota_*^g(\#_\Lambda(\mu_g)) \oplus_{TG} \#_\Lambda(\mu_g), \\ &X_{\alpha(g)} = 0_{T_{g^{-1}}G} \oplus_{TG} (L_g)_*^{\epsilon(\alpha(g))}(X_{\alpha(g)}) \end{aligned}$$

and consequently, using (1.58), (5.1), (5.3), (5.4), Remark 1.1 and the fact that σ is a multiplicative function, we obtain that

$$\begin{aligned} &(d_0\eta)(\epsilon(\alpha(g)))(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\mu_g))), X_{\alpha(g)}) \\ &\qquad = (d_0\eta)(\epsilon(\alpha(g)))(e^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\mu_g))), X_{\alpha(g)}). \end{aligned}$$

On the other hand, from (1.58), *ii)* and Remark 1.1, we deduce that

$$\begin{aligned} &(d_0\eta)(\epsilon(\alpha(g)))(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\mu_g))), \epsilon_*^{\alpha(g)}(Y_{\alpha(g)})) \\ &\qquad = (d_0\eta)(\epsilon(\alpha(g)))(e^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\mu_g))), \epsilon_*^{\alpha(g)}(Y_{\alpha(g)})) = 0, \end{aligned}$$

for $Y_{\alpha(g)} \in T_{\alpha(g)}M$.

The above facts imply that $\epsilon^T(\alpha^T(\#_\Lambda(\mu_g))) = e^{-\sigma(g)}\#_\Lambda(\tilde{\epsilon}(\tilde{\alpha}(\mu_g)))$. In a similar way, one may prove that $\#_\Lambda(\tilde{\epsilon}(\tilde{\beta}(\mu_g))) = \epsilon^T(\beta^T(\#_\Lambda(\mu_g)))$. QED

Using again the results in [23], we have that

Proposition 5.4 *Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid and $\mathfrak{X}_L(G)$ be the set of left-invariant vector fields on G. Denote by (Λ, E) the Jacobi structure on G associated with the contact 1-form η, by $X_0 \in \Gamma(AG)$ the section of the Lie algebroid AG of G satisfying $E = -\overrightarrow{X_0}$ and by $\mathcal{I} : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \mathfrak{X}(G)$ the map defined by*

$$\begin{aligned}\mathcal{I}(\mu_0, f_0) &= \#_\Lambda(e^\sigma \alpha^* \mu_0) - (\alpha^* f_0)\overleftarrow{X_0} \\ &= e^\sigma\Big(\#_\Lambda(\alpha^*\mu_0) - (\alpha^* f_0)(\overrightarrow{X_0} - \#_\Lambda(d_0\sigma))\Big).\end{aligned} \tag{5.5}$$

Then:

i) $\mathcal{I}$ defines an isomorphism of $C^\infty(M, \mathbb{R})$-modules between the spaces $\Omega^1(M) \times C^\infty(M, \mathbb{R})$ and $\mathfrak{X}_L(G)$.

ii) The base manifold M admits a Jacobi structure (Λ_0, E_0) in such a way that the projection β is a Jacobi antimorphism and the pair (α, e^σ) is a conformal Jacobi morphism, that is,

$$\begin{aligned}&\Lambda_0(\alpha(g)) = e^{\sigma(g)}\alpha_*^g(\Lambda(g)), \qquad E_0(\alpha(g)) = \alpha_*^g(\mathcal{H}_{e^\sigma}^{(\Lambda,E)}(g)), \\ &\Lambda_0(\beta(g)) = -\beta_*^g(\Lambda(g)), \qquad E_0(\beta(g)) = -\beta_*^g(E(g)),\end{aligned} \tag{5.6}$$

for all $g \in G$, where $\mathcal{H}_{e^\sigma}^{(\Lambda,E)}$ is the hamiltonian vector field of the function e^σ with respect to the Jacobi structure (Λ, E).

iii) The map $\mathcal{I}$ induces an isomorphism between the Lie algebroids $(T^*M \times \mathbb{R}, [\![\, , \,]\!]_{(\Lambda_0,E_0)}, \widetilde{\#}_{(\Lambda_0,E_0)})$ and AG.

Remark 5.5 Denote also by $\mathcal{I} : T^*M \times \mathbb{R} \to AG$ the Lie algebroid isomorphism induced by the isomorphism of $C^\infty(M, \mathbb{R})$-modules $\mathcal{I} : \Omega^1(M) \times$

$C^\infty(M,\mathbb{R}) \to \mathfrak{X}_L(G)$. Then, from (5.5) and since σ is a multiplicative function, it follows that

$$\mathcal{I}(\mu_x,\gamma) = \#_\Lambda((\alpha_*^{\epsilon(x)})^*(\mu_x)) - \gamma X_0(x), \tag{5.7}$$

for $(\mu_x,\gamma) \in T_x^*M \times \mathbb{R}$, where $(\alpha_*^{\epsilon(x)})^* : T_x^*M \to T_{\epsilon(x)}^*G$ is the adjoint map of the linear map $\alpha_*^{\epsilon(x)} : T_{\epsilon(x)}G \to T_xM$.

Now, let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, there exists a natural right action of the tangent groupoid $TG \rightrightarrows TM$ on the projection $\pi_1 : TM \times \mathbb{R} \to TM$ given by

$$(v_x,\lambda)\cdot X_g = (v_x, X_g(\sigma)+\lambda),$$

for $(v_x,\lambda) \in T_xM \times \mathbb{R}$ and $X_g \in T_gG$ satisfying $\beta^T(X_g) = \pi_1(v_x,\lambda)$ (see Section 1.3.2 for the definition of a right action of a groupoid on a smooth map). The resulting action groupoid is isomorphic to $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ with projections $(\alpha^T)_\sigma$, $(\beta^T)_\sigma$, partial multiplication $\oplus_{TG\times\mathbb{R}}$, inclusion $(\epsilon^T)_\sigma$ and inversion $(\iota^T)_\sigma$ given by

$$\begin{aligned}
&(\alpha^T)_\sigma(X_g,\lambda) = (\alpha^T(X_g), X_g(\sigma)+\lambda), \text{ for } (X_g,\lambda) \in T_gG \times \mathbb{R},\\
&(\beta^T)_\sigma(Y_h,\gamma) = (\beta^T(Y_h),\gamma), \text{ for } (Y_h,\gamma) \in T_hG \times \mathbb{R},\\
&(X_g,\lambda) \oplus_{TG\times\mathbb{R}} (Y_h,\gamma)\\
&\qquad\qquad = (X_g \oplus_{TG} Y_h, \lambda), \text{ if } (\alpha^T)_\sigma(X_g,\lambda) = (\beta^T)_\sigma(Y_h,\gamma),\\
&(\epsilon^T)_\sigma(X_x,\lambda) = (\epsilon^T(X_x),\lambda), \text{ for } (X_x,\lambda) \in T_xM \times \mathbb{R},\\
&(\iota^T)_\sigma(X_g,\lambda) = (\iota^T(X_g), X_g(\sigma)+\lambda), \text{ for } (X_g,\lambda) \in T_gG \times \mathbb{R}.
\end{aligned} \tag{5.8}$$

Now, suppose that $(G \rightrightarrows M, \eta, \sigma)$ is a contact groupoid. Using Remark 1.1, we deduce that the map $\#_{(d_0\eta,\eta)} : TG \times \mathbb{R} \to T^*G \times \mathbb{R}$ given by

$$\#_{(d_0\eta,\eta)}(X_g,\lambda) = (-i_{X_g}(d_0\eta)(g) - \lambda\,\eta(g), \eta(g)(X_g)) \tag{5.9}$$

is an isomorphism of vector bundles. The inverse map of $\#_{(d_0\eta,\eta)}$ is just the homomorphism $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ defined by

$$\#_{(\Lambda,E)}(\mu_g,\gamma) = (\#_\Lambda(\mu_g) + \gamma\, E(g), -\mu_g(E(g))), \tag{5.10}$$

where (Λ, E) is the Jacobi structure associated with the contact 1-form η (see Remark 1.1).

On the other hand, if A^*G is the dual bundle to the Lie algebroid AG then, since $\epsilon(M)$ is a Legendre submanifold of G, the map $\psi_0 : TM \times \mathbb{R} \to A^*G$ given by

$$\psi_0(X_x, \lambda) = (-i_{\epsilon_*^x(X_x)}(d_0\eta)(\epsilon(x)) - \lambda\, \eta(\epsilon(x)))_{|A_xG}, \tag{5.11}$$

for $(X_x, \lambda) \in T_xM \times \mathbb{R}$, is an isomorphism of vector bundles. Note that $\#_{(d_0\eta,\eta)}(\epsilon_*^x(X_x), \lambda) = (\tilde{\epsilon}(\psi_0(X_x, \lambda)), 0)$ and thus the inverse map $\varphi_0 : A^*G \to TM \times \mathbb{R}$ of ψ_0 is defined by

$$\varphi_0(\mu_x) = (\alpha_*^{\epsilon(x)}(\#_\Lambda(\tilde{\epsilon}(\mu_x))), -\mu_x(E(\epsilon(x)) - \epsilon_*^x(\beta_*^{\epsilon(x)}(E(\epsilon(x)))))), \tag{5.12}$$

$\tilde{\epsilon} : A^*G \to T^*G$ being the inclusion of identities in the Lie groupoid $T^*G \rightrightarrows A^*G$.

Next, we consider the maps $\tilde{\alpha}_\sigma, \tilde{\beta}_\sigma : T^*G \times \mathbb{R} \to A^*G$, $\tilde{\epsilon}_\sigma : A^*G \to T^*G \times \mathbb{R}$ and $\tilde{\iota}_\sigma : T^*G \times \mathbb{R} \to T^*G \times \mathbb{R}$ given by

$$\begin{aligned} \tilde{\alpha}_\sigma &= \psi_0 \circ (\alpha^T)_\sigma \circ \#_{(\Lambda,E)}, \quad & \tilde{\beta}_\sigma &= \psi_0 \circ (\beta^T)_\sigma \circ \#_{(\Lambda,E)}, \\ \tilde{\epsilon}_\sigma &= \#_{(d_0\eta,\eta)} \circ (\epsilon^T)_\sigma \circ \varphi_0, \quad & \tilde{\iota}_\sigma &= \#_{(d_0\eta,\eta)} \circ (\iota^T)_\sigma \circ \#_{(\Lambda,E)} \end{aligned} \tag{5.13}$$

and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ defined as follows. If $(\mu_g, \gamma), (\nu_h, \zeta) \in T^*G\times\mathbb{R}$ satisfy $\tilde{\alpha}_\sigma(\mu_g, \gamma) = \tilde{\beta}_\sigma(\nu_h, \zeta)$ then we have that $(\alpha^T)_\sigma(\#_{(\Lambda,E)}(\mu_g, \gamma)) = (\beta^T)_\sigma(\#_{(\Lambda,E)}(\nu_h, \zeta))$, and we may introduce the partial multiplication

$$(\mu_g, \gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h, \zeta) = \#_{(d_0\eta,\eta)}\Big(\#_{(\Lambda,E)}(\mu_g, \gamma) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(\nu_h, \zeta)\Big). \tag{5.14}$$

It is clear $\tilde{\alpha}_\sigma$, $\tilde{\beta}_\sigma$, $\tilde{\epsilon}_\sigma$, $\tilde{\iota}_\sigma$ and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ are the structural functions of a Lie groupoid structure in $T^*G\times\mathbb{R}$ over A^*G. In addition, the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid isomorphism over $\varphi_0 : A^*G \to TM \times \mathbb{R}$.

Lemma 5.6 *If $\tilde{\alpha}$, $\tilde{\beta}$, $\oplus_{T^*G}$, $\tilde{\epsilon}$ and $\tilde{\iota}$ are the structural functions of the Lie groupoid $T^*G \rightrightarrows A^*G$, we have that*

$$\begin{aligned}
&\tilde{\alpha}_\sigma(\mu_g,\gamma) = e^{-\sigma(g)}\tilde{\alpha}(\mu_g), \text{ for } (\mu_g,\gamma) \in T_g^*G \times \mathbb{R},\\
&\tilde{\beta}_\sigma(\nu_h,\zeta) = \tilde{\beta}(\nu_h) - \zeta\,(d_0\sigma)_{\epsilon(\beta(h))|A_{\beta(h)}G}, \text{ for } (\nu_h,\zeta) \in T_h^*G \times \mathbb{R},\\
&\Big((\mu_g,\gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h,\zeta)\Big) = \Big((\mu_g + e^{\sigma(g)}\zeta\,(d_0\sigma)_g) \oplus_{T^*G} (e^{\sigma(g)}\nu_h),\\
&\qquad\qquad \gamma + e^{\sigma(g)}\zeta\Big), \text{ if } \tilde{\alpha}_\sigma(\mu_g,\gamma) = \tilde{\beta}_\sigma(\nu_h,\zeta),\\
&\tilde{\epsilon}_\sigma(\mu_x) = (\tilde{\epsilon}(\mu_x),0), \text{ for } \mu_x \in A_x^*G,\\
&\tilde{\iota}_\sigma(\mu_g,\gamma) = (e^{-\sigma(g)}(\tilde{\iota}(\mu_g) - \gamma(d_0\sigma)_{g^{-1}}), -e^{-\sigma(g)}\gamma), \text{ for } (\mu_g,\gamma) \in T_g^*G \times \mathbb{R}.
\end{aligned} \tag{5.15}$$

Proof: A long computation, using (1.58), (5.1), (5.2), (5.8)-(5.14) and Proposition 5.3, proves the result. QED

Note that the maps $\tilde{\alpha}_\sigma$, $\tilde{\beta}_\sigma$, $\tilde{\epsilon}_\sigma$, $\tilde{\iota}_\sigma$ and the partial multiplication $\oplus_{T^*G\times\mathbb{R}}$ do not depend on the contact 1-form η. In fact, one may prove the following result.

Theorem 5.7 *Let $G \rightrightarrows M$ be an arbitrary Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then:*

i) The product manifold $T^*G \times \mathbb{R}$ admits a Lie groupoid structure over A^*G with structural functions given by (5.15).

ii) If η_G is the canonical contact 1-form on $T^*G\times\mathbb{R}$ and $\bar{\pi}_G : T^*G\times\mathbb{R} \to G$ is the canonical projection then $(T^*G\times\mathbb{R} \rightrightarrows A^*G, \eta_G, \sigma\circ\bar{\pi}_G)$ is a contact groupoid.

Proof: Since σ is a multiplicative function, we obtain that

$$\epsilon^*\sigma = 0. \tag{5.16}$$

Moreover, if $(g,h) \in G^{(2)}$ and $\alpha(g) = \beta(h) = x \in M$ then, from (1.58), it follows that

$$\begin{aligned}
&\tilde{\alpha}((d_0\sigma)(g)) = \tilde{\beta}((d_0\sigma)(h)) = (d_0\sigma)(\epsilon(x))_{|A_xG},\\
&(d_0\sigma)(gh) = (d_0\sigma)(g) \oplus_{T^*G} (d_0\sigma)(h).
\end{aligned} \tag{5.17}$$

In addition, using again (1.58) and the fact that σ is a multiplicative function, we have that

$$\tilde{\epsilon}((d_0\sigma)(\epsilon(x))_{|A_xG}) = (d_0\sigma)(\epsilon(x)), \quad \tilde{\iota}((d_0\sigma)(g)) = (d_0\sigma)(g^{-1}), \tag{5.18}$$

for $x \in M$ and $g \in G$.
Thus, from (5.15)-(5.18), we deduce *i)*.
Now, let $G \times \mathbb{R} \rightrightarrows M$ be the semi-direct Lie groupoid with projections α', β', partial multiplication m', inclusion ϵ' and inversion ι' defined by

$$\begin{aligned}
&\alpha'(g,\gamma) = \alpha(g), \text{ for } (g,\gamma) \in G \times \mathbb{R},\\
&\beta'(h,\zeta) = \beta(h), \text{ for } (h,\zeta) \in G \times \mathbb{R},\\
&m'((g,\gamma),(h,\zeta)) = (gh, \gamma + e^{\sigma(g)}\zeta), \text{ if } \alpha'(g,\gamma) = \beta'(h,\zeta),\\
&\epsilon'(x) = (\epsilon(x), 0), \text{ for } x \in M,\\
&\iota'(g,\gamma) = (\iota(g), -e^{-\sigma(g)}\gamma), \text{ for } (g,\gamma) \in G \times \mathbb{R}.
\end{aligned} \tag{5.19}$$

Using (5.19), one may prove that the partial multiplication $\oplus_{T(G\times\mathbb{R})}$ in the tangent Lie groupoid $T(G \times \mathbb{R}) \rightrightarrows TM$ is given by

$$\begin{aligned}
&\Big(X_g + \psi\frac{\partial}{\partial t}_{|\gamma}\Big) \oplus_{T(G\times\mathbb{R})} \Big(Y_h + \varphi\frac{\partial}{\partial t}_{|\zeta}\Big)\\
&\qquad = (X_g \oplus_{TG} Y_h) + (\psi + e^{\sigma(g)}(\zeta X_g(\sigma) + \varphi))\frac{\partial}{\partial t}_{|\gamma + e^{\sigma(g)}\zeta}.
\end{aligned} \tag{5.20}$$

Next, we consider the map $\tilde{\pi}_G : T^*G \times \mathbb{R} \to G \times \mathbb{R}$ given by

$$\tilde{\pi}_G(\mu_g, \gamma) = (\pi_G(\mu_g), \gamma),$$

for $(\mu_g, \gamma) \in T^*_gG \times \mathbb{R}$, where $\pi_G : T^*G \to G$ is the canonical projection. From (5.15) and (5.19), we deduce that $\tilde{\pi}_G$ is a Lie groupoid morphism over the map $\tilde{\pi}_0 : A^*G \to M$ defined by

$$\tilde{\pi}_0(\mu_x) = x,$$

for $\mu_x \in A^*_xG$. Therefore, the tangent map to $\tilde{\pi}_G$, $T\tilde{\pi}_G : T(T^*G \times \mathbb{R}) \to T(G \times \mathbb{R})$, given by

$$T\tilde{\pi}_G(X_{\mu_g} + \psi\frac{\partial}{\partial t}_{|\gamma}) = (\pi_G)_*^{\mu_g}(X_{\mu_g}) + \psi\frac{\partial}{\partial t}_{|\gamma}, \tag{5.21}$$

for $X_{\mu_g}+\psi\frac{\partial}{\partial t}_{|\gamma}\in T_{(\mu_g,\gamma)}(G\times\mathbb{R})$, is also a Lie groupoid morphism (over the map $T\tilde{\pi}_0: T(A^*G)\to TM$) between the tangent Lie groupoids $T(T^*G\times\mathbb{R})\rightrightarrows T(A^*G)$ and $T(G\times\mathbb{R})\rightrightarrows TM$.
On the other hand, if η_G is the canonical contact 1-form on $T^*G\times\mathbb{R}$ then, using (2.10) and (5.21), we have that

$$\begin{aligned}\eta_G(\mu_g,\lambda)(X_{\mu_g}+\psi\tfrac{\partial}{\partial t}_{|\gamma}) &= -\lambda_{T^*G}(\mu_g)(X_{\mu_g})+d_0t_{|\gamma}(\psi\tfrac{\partial}{\partial t}_{|\gamma})\\ &= -\mu_g((\pi_G)_*^{\mu_g}(X_{\mu_g}))+\psi\\ &= (-\mu_g+d_0t_{|\gamma})(T\tilde{\pi}_G(X_{\mu_g}+\psi\tfrac{\partial}{\partial t}_{|\gamma})).\end{aligned} \tag{5.22}$$

Thus, using (5.15), (5.20), (5.21), (5.22) and the fact that $T\tilde{\pi}_G$ is a Lie groupoid morphism, we conclude that

$$\eta_G((\mu_g,\gamma)\oplus_{T^*G\times\mathbb{R}}(\nu_h,\zeta))=\eta_G(\mu_g,\gamma)\oplus_{T^*(T^*G\times\mathbb{R})}(e^{\sigma(g)}\eta_G(\nu_h,\zeta)),$$

that is, $(T^*G\times\mathbb{R}\rightrightarrows A^*G,\eta_G,\bar{\sigma})$ is a contact groupoid, where $\bar{\sigma}\in C^\infty(T^*G\times\mathbb{R})$ is the function given by $\bar{\sigma}=\sigma\circ\bar{\pi}_G$. QED

Remark 5.8 *i)* Let $G\rightrightarrows M$ be a Lie groupoid and $\sigma: G\to\mathbb{R}$ be a multiplicative function. Using the multiplicative function σ, one may introduce the Lie groupoid structure in T^*G over A^*G with structural functions $\tilde{\alpha}^*_\sigma$, $\tilde{\beta}^*_\sigma$, $\oplus^\sigma_{T^*G}$, $\tilde{\epsilon}^*_\sigma$ and $\tilde{\iota}^*_\sigma$ given by

$$\begin{aligned}&\tilde{\alpha}^*_\sigma(\mu_g)=e^{-\sigma(g)}\tilde{\alpha}(\mu_g),\ \text{for } \mu_g\in T^*_gG,\\ &\tilde{\beta}^*_\sigma(\nu_h)=\tilde{\beta}(\nu_h),\ \text{for } \nu_h\in T^*_hG,\\ &(\mu_g\oplus^\sigma_{T^*G}\nu_h)=\mu_g\oplus_{T^*G}(e^{\sigma(g)}\nu_h),\ \text{if } \tilde{\alpha}^*_\sigma(\mu_g)=\tilde{\beta}^*_\sigma(\nu_h),\\ &\tilde{\epsilon}^*_\sigma(\mu_x)=\tilde{\epsilon}(\mu_x),\ \text{for } \mu_x\in A^*_xG,\\ &\tilde{\iota}^*_\sigma(\mu_g)=e^{-\sigma(g)}\tilde{\iota}(\mu_g),\ \text{for } \mu_g\in T^*_gG.\end{aligned} \tag{5.23}$$

We call this Lie groupoid the *σ-cotangent groupoid*.

In fact, if we consider on $T^*G\times\mathbb{R}$ the Lie groupoid structure over A^*G with structural functions defined by (5.15) then the canonical inclusion

$$T^*G\to T^*G\times\mathbb{R},\quad \mu_g\in T^*_gG\mapsto(\mu_g,0)\in T^*_gG\times\mathbb{R},$$

is a Lie groupoid monomorphism over the identity of A^*G.

ii) Let $G \rightrightarrows M$ be a Lie groupoid, $\sigma : G \to \mathbb{R}$ be a multiplicative function and $TG\times\mathbb{R} \rightrightarrows TM\times\mathbb{R}$, $T^*G\times\mathbb{R} \rightrightarrows A^*G$ be the corresponding Lie groupoids with structural functions given by (5.8) and (5.15). If σ identically vanishes then we recover, by projection, the tangent and cotangent Lie groupoids $TG \rightrightarrows TM$ and $T^*G \rightrightarrows A^*G$ (see (1.55) and (1.58)).

Remark 5.9 *i)* A Lie groupoid $G \rightrightarrows M$ is said to be *symplectic* if G admits a symplectic 2-form Ω in such a way that the graph of the partial multiplication in G is a Lagrangian submanifold of the symplectic manifold $(G\times G\times G, \Omega\oplus\Omega\oplus(-\Omega))$ (see [14]). If $G \rightrightarrows M$ is an arbitrary Lie groupoid with Lie algebroid AG and on the cotangent Lie groupoid T^*G we consider the canonical symplectic 2-form $\Omega_{T^*G} = -d_0\lambda_{T^*G}$ then T^*G is a symplectic groupoid over A^*G (see [14]).

ii) Let $G \rightrightarrows M$ be a symplectic groupoid with exact symplectic 2-form $\Omega = -d_0\lambda$. Then, since $\mathbb{R}$ is a Lie group, the product manifold $G\times\mathbb{R}$ is a Lie groupoid over M (see Example 3 in Section 1.3.2). In addition, $(G\times\mathbb{R} \rightrightarrows M, \eta, 0)$ is a contact groupoid, where η is the 1-form on $G\times\mathbb{R}$ given by $\eta = \pi_2^*(d_0t) - \pi_1^*(\lambda)$, and $\pi_1 : G\times\mathbb{R} \to G$, $\pi_2 : G\times\mathbb{R}\to\mathbb{R}$ are the canonical projections (see [70]). In particular, if $G \rightrightarrows M$ is an arbitrary Lie groupoid with Lie algebroid AG then we have that $(T^*G\times\mathbb{R} \rightrightarrows A^*G, \eta_G, 0)$ is a contact groupoid, η_G being the canonical contact 1-form on $T^*G\times\mathbb{R}$. Note that, using Theorem 5.7, we directly deduce this result.

Let $G \rightrightarrows M$ be an arbitrary Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. From Proposition 5.4, it follows that the contact groupoid structure on $T^*G\times\mathbb{R}$ induces a Jacobi structure on the vector bundle A^*G. Next, we will describe such a Jacobi structure. In fact, we will show the Jacobi structure on A^*G is the homogeneous Jacobi structure associated with a Jacobi algebroid structure on AG. We recall that a Jacobi algebroid structure $((\llbracket\, , \,\rrbracket, \rho), \phi_0)$ on a real vector bundle $A \to M$ induces a homogeneous Jacobi structure $(\Lambda_{(A^*,\phi_0)}, E_{(A^*,\phi_0)})$ on the dual bundle to A (see Theorem 2.7).

Theorem 5.10 *Let $G \rightrightarrows M$ be a Lie groupoid with Lie algebroid AG and $\sigma : G \to \mathbb{R}$ be a multiplicative function. If $\bar{\pi}_G : T^*G \times \mathbb{R} \to G$ is the canonical projection, η_G is the canonical contact 1-form on $T^*G \times \mathbb{R}$ and (Λ_0, E_0) is the Jacobi structure on A^*G induced by the contact groupoid $(T^*G \times \mathbb{R} \rightrightarrows A^*G, \eta_G, \bar{\sigma} = \sigma \circ \bar{\pi}_G)$ then*

$$\Lambda_0 = \Lambda_{(A^*G,\phi_0)}, \quad E_0 = E_{(A^*G,\phi_0)}, \tag{5.24}$$

where $\phi_0 \in \Gamma(A^*G)$ is the 1-cocycle of the Lie algebroid AG defined by

$$\phi_0(x)(X_x) = X_x(\sigma), \tag{5.25}$$

for $x \in M$ and $X_x \in A_xG$.

Proof: Denote by $\pi_1 : T^*G \times \mathbb{R} \to T^*G$ the canonical projection onto the first factor. It is easy to prove that π_1 is a Jacobi morphism between the contact manifold $(T^*G \times \mathbb{R}, \eta_G)$ and the symplectic manifold (T^*G, Ω_{T^*G}). Thus,

$$\{f \circ \pi_1, g \circ \pi_1\}_{\eta_G} = \{f, g\}_{\Omega_{T^*G}} \circ \pi_1, \tag{5.26}$$

for $f, g \in C^\infty(T^*G, \mathbb{R})$, $\{\, , \,\}_{\eta_G}$ (respectively, $\{\, , \,\}_{\Omega_{T^*G}}$) being the Jacobi bracket (respectively, Poisson bracket) associated with the contact 1-form η_G (respectively, the symplectic 2-form Ω_{T^*G}).

Now, suppose that $\{\, , \,\}_{(\Lambda_0,E_0)}$ is the Jacobi bracket associated with the Jacobi structure (Λ_0, E_0). From (5.6), it follows that

$$\tilde{\alpha}_\sigma^*\{f_{A^*G}, g_{A^*G}\}_{(\Lambda_0,E_0)} = e^{-\bar{\sigma}}\{e^{\bar{\sigma}}\tilde{\alpha}_\sigma^* f_{A^*G}, e^{\bar{\sigma}}\tilde{\alpha}_\sigma^* g_{A^*G}\}_{\eta_G} \tag{5.27}$$

for $f_{A^*G}, g_{A^*G} \in C^\infty(A^*G, \mathbb{R})$. Thus, if $X, Y \in \Gamma(AG)$ and $\tilde{X}, \tilde{Y}$ are the corresponding linear functions on A^*G, then (see (5.15), (5.26) and (5.27))

$$\begin{array}{rcl} \{\tilde{X}, \tilde{Y}\}_{(\Lambda_0,E_0)}(\tilde{\alpha}_\sigma(\mu_g, \gamma)) & = & (e^{-\bar{\sigma}}\{\tilde{\alpha}^*(\tilde{X}) \circ \pi_1, \tilde{\alpha}^*(\tilde{Y}) \circ \pi_1\}_{\eta_G})(\mu_g, \gamma) \\ & = & e^{-\sigma(g)}\{\tilde{\alpha}^*(\tilde{X}), \tilde{\alpha}^*(\tilde{Y})\}_{\Omega_{T^*G}}(\mu_g), \end{array} \tag{5.28}$$

for $(\mu_g, \gamma) \in T_g^*G \times \mathbb{R}$. On the other hand, using the results in [14], we have that

$$(\pi_G)_*^{\nu_h}(\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{X})}(\nu_h)) = \overleftarrow{X}(h), \quad (\pi_G)_*^{\nu_h}(\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{Y})}(\nu_h)) = \overleftarrow{Y}(h), \tag{5.29}$$

for $h \in G$ and $\nu_h \in T^*_h G$, where $\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{X})}$ (respectively, $\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{Y})}$) is the hamiltonian vector field of the function $\tilde{\alpha}^*(\tilde{X})$ (respectively, $\tilde{\alpha}^*(\tilde{Y})$) with respect to the symplectic structure Ω_{T^*G}. Therefore, $(\mathcal{L}_0)_{\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{X})}}\lambda_{T^*G} = (\mathcal{L}_0)_{\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{Y})}}\lambda_{T^*G}$ $= 0$ and from (5.28) and (5.29), we conclude that

$$\begin{array}{rcl} \{\tilde{X},\tilde{Y}\}_{(\Lambda_0,E_0)}(\tilde{\alpha}_\sigma(\mu_g,\gamma)) & = & e^{-\sigma(g)}\lambda_{T^*G}(\mu_g)\Big([\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{X})},\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{Y})}](\mu_g)\Big) \\ & = & e^{-\sigma(g)}\mu_g(\overleftarrow{[\![X,Y]\!]}(g)) \\ & = & \tilde{\alpha}_\sigma(\mu_g,\gamma)([\![X,Y]\!](\alpha(g))), \end{array}$$

$[\![\;,\;]\!]$ being the Lie bracket on AG. Consequently,

$$\{\tilde{X},\tilde{Y}\}_{(\Lambda_0,E_0)} = \widetilde{[\![X,Y]\!]}. \tag{5.30}$$

Next, we will show that

$$\{\tilde{X},1\}_{(\Lambda_0,E_0)} = \phi_0(X)\circ\tau^*, \tag{5.31}$$

where $\tau^* : A^*G \to M$ is the bundle projection. Using (5.15), (5.26) and (5.27), it follows that

$$\begin{array}{rcl} \{\tilde{X},1\}_{(\Lambda_0,E_0)}(\tilde{\alpha}_\sigma(\mu_g,\gamma)) & = & (e^{-\bar{\sigma}}\{\tilde{\alpha}^*(\tilde{X})\circ\pi_1, e^{\sigma\circ\pi_G}\circ\pi_1\}_{\eta_G})(\mu_g,\gamma) \\ & = & e^{-\sigma(g)}\{\tilde{\alpha}^*(\tilde{X}), e^{\sigma\circ\pi_G}\}_{\Omega_{T^*G}}(\mu_g) \\ & = & (\pi_G)^{\mu_g}_*(\mathcal{H}^{\Omega_{T^*G}}_{\tilde{\alpha}^*(\tilde{X})}(\mu_g))(\sigma). \end{array}$$

Thus, from (5.25) and (5.29), we obtain that

$$\{\tilde{X},1\}_{(\Lambda_0,E_0)}(\tilde{\alpha}_\sigma(\mu_g,\gamma)) = (\phi_0(X)\circ\tau^*)(\tilde{\alpha}_\sigma(\mu_g,\gamma)).$$

This implies that (5.31) holds.

Finally, using (5.30), (5.31) and Remark 2.8, we deduce (5.24). *QED*

5.2 Jacobi groupoids: definition and characterization

Motivated by the results obtained in Section 5.1 about contact groupoids, we introduce the following definition.

Definition 5.11 *Let $G \rightrightarrows M$ be a Lie groupoid, (Λ, E) be a Jacobi structure on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a* Jacobi groupoid *if the homomorphism $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ given by*

$$\#_{(\Lambda,E)}(\mu_g, \gamma) = (\#_\Lambda(\mu_g) + \gamma\, E(g), -\mu_g(E(g))) \tag{5.32}$$

is a morphism of Lie groupoids over some map $\varphi_0 : A^*G \to TM \times \mathbb{R}$, where the structural functions of the Lie groupoid structure on $T^*G \times \mathbb{R} \rightrightarrows A^*G$ (respectively, $TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$) are given by (5.15) (respectively, (5.8)).

Remark 5.12 Since $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a morphism of Lie groupoids, we deduce that

$$\varphi_0 = (\alpha^T)_\sigma \circ \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma = (\beta^T)_\sigma \circ \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma.$$

Thus, if $\mu_x \in A^*_xG$, it follows that

$$\varphi_0(\mu_x) = \Big(\alpha_*^{\epsilon(x)}(\#_\Lambda(\tilde{\epsilon}(\mu_x))), -\mu_x\left(E(\epsilon(x)) - \epsilon_*^x(\beta_*^{\epsilon(x)}(E(\epsilon(x))))\right)\Big). \tag{5.33}$$

A characterization of a Jacobi groupoid is the following one.

Theorem 5.13 *Let $G \rightrightarrows M$ be a Lie groupoid, (Λ, E) be a Jacobi structure on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a* Jacobi groupoid *if and only if the following conditions hold:*

i) The map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism over some map $\tilde{\varphi}_0 : A^*G \to TM$ from the σ-cotangent groupoid $T^*G \rightrightarrows A^*G$ to the tangent Lie groupoid $TG \rightrightarrows TM$.

ii) E is a right-invariant vector field on G and $E(\sigma) = 0$.

iii) If $X_0 \in \Gamma(AG)$ is the section of the Lie algebroid AG satisfying $E = -\overrightarrow{X_0}$, we have that

$$\#_\Lambda(d_0\sigma) = \overrightarrow{X_0} - e^{-\sigma}\overleftarrow{X_0}. \tag{5.34}$$

Proof: Suppose that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

If $\mu_g \in T_g^*G$ and $\nu_h \in T_h^*G$ satisfy $\tilde{\alpha}_\sigma^*(\mu_g) = \tilde{\beta}_\sigma^*(\nu_h)$ then, from (5.8), (5.15), (5.23), (5.32) and since the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid homomorphism, we have that $\alpha^T(\#_\Lambda(\mu_g)) = \beta^T(\#_\Lambda(\nu_h))$ and

$$\#_\Lambda(\mu_g \oplus_{T^*G}^\sigma \nu_h) = \#_\Lambda(\mu_g) \oplus_{TG} \#_\Lambda(\nu_h).$$

This proves *i)*.

On the other hand, we have that

$$(\alpha^T)_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\alpha}_\sigma, \qquad (\beta^T)_\sigma \circ \#_{(\Lambda,E)} = \varphi_0 \circ \tilde{\beta}_\sigma. \tag{5.35}$$

Thus, if $g \in G$, it follows that

$$((\alpha^T)_\sigma \circ \#_{(\Lambda,E)})(0_{T_g^*G}, 1) = (\varphi_0 \circ \tilde{\alpha}_\sigma)(0_{T_g^*G}, 1),$$

which, using (5.8), (5.15) and (5.32), implies that

$$\alpha_*^g(E(g)) = 0, \qquad E(g)(\sigma) = 0. \tag{5.36}$$

Now, if $(g,h) \in G^{(2)}$ then, from (5.15), we deduce that

$$\tilde{\alpha}_\sigma(0_{T_g^*G}, 1) = \tilde{\beta}_\sigma(0_{T_h^*G}, 0) = 0$$

and therefore

$$\begin{aligned}\#_{(\Lambda,E)}&((0_{T_g^*G}, 1) \oplus_{T^*G\times\mathbb{R}} (0_{T_h^*G}, 0)) \\ &= \#_{(\Lambda,E)}(0_{T_g^*G}, 1) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(0_{T_h^*G}, 0).\end{aligned}$$

Consequently, using (5.8), (5.15) and (5.32), we obtain that

$$(R_h)_*^g(E(g)) = E(gh). \tag{5.37}$$

This proves *ii)* (see (5.36) and (5.37)).

Next, we will show that (5.34) holds.
From (5.35), it follows that

$$((\beta^T)_\sigma \circ \#_{(\Lambda,E)})(e^{\sigma(g)}(d_0\sigma)(g), e^{\sigma(g)}) = (\varphi_0 \circ \tilde{\beta}_\sigma)(e^{\sigma(g)}(d_0\sigma)(g), e^{\sigma(g)})$$

and thus, using (5.8), (5.15), (5.17) and (5.32), we have that

$$\beta_*^g(\mathcal{H}_{e^\sigma}^{(\Lambda,E)}(g)) = 0, \tag{5.38}$$

where $\mathcal{H}_{e^\sigma}^{(\Lambda,E)}$ is the hamiltonian vector field of the function e^σ with respect to the Jacobi structure (Λ, E).

On the other hand, suppose that $(g,h) \in G^{(2)}$. Then, from (5.15) and (5.17), we deduce that

$$\tilde{\alpha}_\sigma(0_{T_g^*G}, 0) = \tilde{\beta}_\sigma(e^{\sigma(h)}(d_0\sigma)(h), e^{\sigma(h)}) = 0$$

and therefore

$$\begin{aligned} &\#_{(\Lambda,E)}((0_{T_g^*G},0) \oplus_{T^*G\times\mathbb{R}} (e^{\sigma(h)}(d_0\sigma)(h), e^{\sigma(h)})) \\ &\qquad = \#_{(\Lambda,E)}(0_{T_g^*G},0) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(e^{\sigma(h)}(d_0\sigma)(h), e^{\sigma(h)}). \end{aligned}$$

Consequently, using (5.8), (5.15), (5.17), (5.32), the fact that σ is multiplicative and since $E(\sigma) = 0$, we obtain that

$$(L_g)_*^h(\mathcal{H}_{e^\sigma}^{(\Lambda,E)}(h)) = \mathcal{H}_{e^\sigma}^{(\Lambda,E)}(gh). \tag{5.39}$$

Now, if x is a point of M then, using (1.58), it follows that the map

$$\tilde{\epsilon}_{|A_x^*G} : A_x^*G \to T_{\epsilon(x)}^*G$$

is a linear isomorphism between the vector spaces A_x^*G and the annihilator of the subspace $T_{\epsilon(x)}\epsilon(M)$, that is, $(T_{\epsilon(x)}\epsilon(M))^\circ$. Thus, from (1.55), (1.58), (5.8), (5.15), (5.33) and since $(\epsilon^T)_\sigma \circ \varphi_0 = \#_{(\Lambda,E)} \circ \tilde{\epsilon}_\sigma$, we conclude that

$$\#_\Lambda(T_{\epsilon(x)}\epsilon(M))^\circ \subseteq T_{\epsilon(x)}\epsilon(M), \tag{5.40}$$

for all $x \in M$. This implies that $\#_\Lambda(d_0\sigma)(\epsilon(x)) \in T_{\epsilon(x)}\epsilon(M)$ (note that $(d_0\sigma)(\epsilon(x)) \in (T_{\epsilon(x)}\epsilon(M))^\circ$) or, equivalently,

$$\#_\Lambda((d_0\sigma)(\epsilon(x))) = \epsilon_*^x(\beta_*^{\epsilon(x)}(\#_\Lambda(d_0\sigma)(\epsilon(x)))).$$

But, $\mathcal{H}_{e^\sigma}^{(\Lambda,E)}$ is β-vertical and therefore

$$\beta_*^{\epsilon(x)}(\#_\Lambda((d_0\sigma)(\epsilon(x)))) = -\beta_*^{\epsilon(x)}(E(\epsilon(x))),$$

that is,

$$\mathcal{H}_{e^\sigma}^{(\Lambda,E)}(\epsilon(x)) = \#_\Lambda((d_0\sigma)(\epsilon(x))) + E(\epsilon(x)) = -\overleftarrow{X}_0(\epsilon(x)). \tag{5.41}$$

Consequently, using (5.38), (5.39) and (5.41), we deduce that $\#_\Lambda(d_0\sigma) = \overrightarrow{X}_0 - e^{-\sigma}\overleftarrow{X}_0$.

Conversely, assume that *i), ii)* and *iii)* hold.

We must prove that the map $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid morphism or, equivalently, that if $(\mu_g, \gamma) \in T_g^*G \times \mathbb{R}$ and $(\nu_h, \zeta) \in T_h^*G \times \mathbb{R}$ satisfy $\tilde{\alpha}(\mu_g, \gamma) = \tilde{\beta}(\nu_h, \zeta)$ then

$$(\alpha^T)_\sigma(\#_{(\Lambda,E)}(\mu_g, \gamma)) = (\beta^T)_\sigma(\#_{(\Lambda,E)}(\nu_h, \zeta)), \tag{5.42}$$

$$\#_{(\Lambda,E)}((\mu_g, \gamma) \oplus_{T^*G\times\mathbb{R}} (\nu_h, \zeta)) = \#_{(\Lambda,E)}(\mu_g, \gamma) \oplus_{TG\times\mathbb{R}} \#_{(\Lambda,E)}(\nu_h, \zeta). \tag{5.43}$$

Now, from (5.15), (5.17) and (5.23), we obtain that

$$\tilde{\alpha}^*_\sigma(\mu_g + \zeta\, e^{\sigma(g)}(d_0\sigma)(g)) = \tilde{\beta}^*_\sigma(\nu_h)$$

which, using *i)*, implies that

$$\alpha^T(\#_\Lambda(\mu_g + \zeta\, e^{\sigma(g)}(d_0\sigma)(g))) = \beta^T(\#_\Lambda(\nu_h)), \tag{5.44}$$

$$\begin{aligned} &\#_\Lambda((\mu_g + \zeta\, e^{\sigma(g)}(d_0\sigma)(g)) \oplus_{T^*G} e^{\sigma(g)}\nu_h) \\ &\qquad = \#_\Lambda(\mu_g + \zeta\, e^{\sigma(g)}(d_0\sigma)(g)) \oplus_{TG} \#_\Lambda(e^{\sigma(g)}\nu_h). \end{aligned} \tag{5.45}$$

Thus, form (5.43), *ii)* and *iii)*, it follows that

$$\begin{aligned} \alpha^T(\#_\Lambda(\mu_g)) &= \beta^T(\#_\Lambda(\nu_h)) + \zeta\alpha^T(\overleftarrow{X}_0(g)) \\ &= \beta^T(\#_\Lambda(\nu_h) + \zeta E(h)). \end{aligned} \tag{5.46}$$

Moreover, using *ii)*, the fact that σ is multiplicative and since $(\tilde{\alpha}_\sigma(\mu_g, \gamma))(X_0(\alpha(g))) = (\tilde{\beta}_\sigma(\nu_h, \zeta))(X_0(\beta(h)))$, we deduce that

$$e^{-\sigma(g)}\mu_g(\overleftarrow{X}_0(g)) = -\nu_h(E(h)). \tag{5.47}$$

From (5.8), (5.36), (5.46) and (5.47), we conclude that (5.42) holds.

On the other hand, using (1.58), (5.15), (5.32) and *ii)*, we obtain that

$$\begin{aligned}
&\#_{(\Lambda,E)}((\mu_g,\gamma)\oplus_{T^*G\times\mathbb{R}}(\nu_h,\zeta))\\
&\quad=(\#_\Lambda(\mu_g+\zeta\, e^{\sigma(g)}(d_0\sigma)(g))\oplus_{TG}\#_\Lambda(e^{\sigma(g)}\nu_h)\\
&\qquad+(\gamma+e^{\sigma(g)}\zeta)(E(g)\oplus_{TG}0_{T_hG}),-((\mu_g+e^{\sigma(g)}\zeta(d_0\sigma)(g))\\
&\qquad\oplus_{T^*G}(e^{\sigma(g)}\nu_h))(E(g)\oplus_{TG}0_{T_hG}))\\
&\quad=(\#_\Lambda(\mu_g+\zeta e^{\sigma(g)}(d_0\sigma)(g))\oplus_{TG}\#_\Lambda(e^{\sigma(g)}\nu_h)\\
&\qquad+(\gamma+e^{\sigma(g)}\zeta)(E(g)\oplus_{TG}0_{T_hG}),-\mu_g(E(g))).
\end{aligned}$$

Furthermore, if $\mu'_g\in T^*_gG$ and $\nu'_h\in T^*_hG$ satisfy $\tilde\alpha(\mu'_g)=\tilde\beta(\nu'_h)$ then, from (1.58) and *iii)*, we have that

$$\begin{aligned}
(\mu'_g\oplus_{T^*G}\nu'_h)&(\#_\Lambda(\mu_g+\zeta\, e^{\sigma(g)}(d_0\sigma)(g))\oplus_{TG}\#_\Lambda(e^{\sigma(g)}\nu_h)\\
&+(\gamma+e^{\sigma(g)}\zeta)(E(g)\oplus_{TG}0_{T_hG}))=-\tilde\alpha(\mu'_g)(X_0(\alpha(g))).
\end{aligned}$$

Finally, using (1.58) and *ii)*, it follows that

$$(\mu'_g\oplus_{T^*G}\nu'_h)((\#_\Lambda(\mu_g)+\gamma E(g))\oplus_{TG}(\#_\Lambda(\nu_h)+\zeta E(h)))=-\tilde\beta(\nu'_h)(X_0(\beta(h))).$$

Therefore, we conclude that

$$\begin{aligned}
\#_\Lambda(\mu_g+\zeta\, e^{\sigma(g)}(d_0\sigma)(g))\oplus_{TG}\#_\Lambda(e^{\sigma(g)}\nu_h)+(\gamma+e^{\sigma(g)}\zeta)(E(g)\oplus_{TG}0_{T_hG})\\
=(\#_\Lambda(\mu_g)+\gamma E(g))\oplus_{TG}(\#_\Lambda(\nu_h)+\zeta E(h))
\end{aligned}\tag{5.48}$$

and thus, from (5.8), (5.32) and (5.48), we deduce (5.43). *QED*

Remark 5.14 Let $(G\rightrightarrows M,\Lambda,E,\sigma)$ be a Jacobi groupoid. Then, by Theorem 5.13, we have that $\#_\Lambda:T^*G\to TG$ is a Lie groupoid morphism from the σ-cotangent groupoid $T^*G\rightrightarrows A^*G$ to the tangent Lie groupoid $TG\rightrightarrows TM$. As a consequence, we get that

$$e^{-\sigma}\#_\Lambda\circ\tilde\epsilon\circ\tilde\alpha=\epsilon^T\circ\alpha^T\circ\#_\Lambda,\quad \#_\Lambda\circ\tilde\epsilon\circ\tilde\beta=\epsilon^T\circ\beta^T\circ\#_\Lambda,$$

$\tilde\alpha$, $\tilde\beta$ and $\tilde\epsilon$ being the projections and the inclusion of the cotangent Lie groupoid $T^*G\rightrightarrows A^*G$.

Some other basic properties of Jacobi groupoids, different from the ones we obtained in Theorem 5.13, are shown in the following result.

Proposition 5.15 *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then:*
i) $M \cong \epsilon(M)$ is a coisotropic submanifold in G.
ii) If g and h are elements of G such that $\alpha(g) = \beta(h) = x$ and $\mathcal{X}$ and $\mathcal{Y}$ are (local) bisections through the points g and h, $\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$, then

$$\begin{aligned}\Lambda(gh) &= (R_{\mathcal{Y}})_*^g(\Lambda(g)) + e^{-\sigma(g)}(L_{\mathcal{X}})_*^h(\Lambda(h)) \\ &\quad - e^{-\sigma(g)}(L_{\mathcal{X}} \circ R_{\mathcal{Y}})_*^{\epsilon(x)}(\Lambda(\epsilon(x))).\end{aligned} \tag{5.49}$$

Proof: Using (5.40), we obtain that M is a coisotropic submanifold in G.

Next, we will prove *ii)*. Let Ξ be the 2-vector on $G \times G \times G$ defined by $\Xi(g,h,k) = e^{\sigma(g)}\Lambda(g) + \Lambda(h) - e^{\sigma(g)}\Lambda(k)$. Then, since the map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism from the σ-cotangent groupoid $T^*G \rightrightarrows A^*G$ to the tangent Lie groupoid $TG \rightrightarrows TM$ (see Theorem 5.13), it follows that the graph of the multiplication in G, $\{(g,h,gh) \in G \times G \times G \,/\, \alpha(g) = \beta(h)\}$, is a coisotropic submanifold of $G \times G \times G$ with respect to Ξ.

Now, denote by $AD(G)$ the affinoid diagram corresponding to the Lie groupoid G, that is (see [118]),

$$AD(G) = \{(k,g,h,r) \in G \times G \times G \times G \,/\, \alpha(h) = \alpha(k),\ \beta(k) = \beta(g),\ r = hk^{-1}g\}.$$

Then, following the proof of Theorem 4.5 in [118], we obtain that $AD(G)$ is a coisotropic submanifold of $G \times G \times G \times G$ with respect to the 2-vector $\widetilde{\widetilde{\Xi}}$ given by

$$\widetilde{\widetilde{\Xi}}(k,g,h,r) = e^{\sigma(k)}\Lambda(k) - e^{\sigma(k)}\Lambda(g) - e^{\sigma(h)}\Lambda(h) + e^{\sigma(h)}\Lambda(r).$$

On the other hand, if g and h are elements of G satisfying $\alpha(g) = \beta(h) = x$, we have that $(gh, g, h, \epsilon(x))$ is an element of $AD(G)$. In addition, for any $\xi \in T^*_{gh}G$ and $\mathcal{X}, \mathcal{Y}$ (local) bisections of G through the points g and h ($\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$), it follows from Lemma 2.6 in [120] that

$$(-\xi, ((R_{\mathcal{Y}})_*^g)^*(\xi), ((L_{\mathcal{X}})_*^h)^*(\xi), -((R_{\mathcal{Y}} \circ L_{\mathcal{X}})_*^{\epsilon(x)})^*(\xi))$$

is a conormal vector to $AD(G)$ at $(gh, g, h, \epsilon(x))$, i.e., it is an element of $(T_{(gh,g,h,\epsilon(x))}AD(G))^\circ$. Here, $R_{\mathcal{Y}}$ and $L_{\mathcal{X}}$ denote the right-translation and the left-translation induced by $\mathcal{Y}$ and $\mathcal{X}$ (see (1.51)). Therefore, if $\xi, \eta \in T^*_{gh}G$, we deduce that

$$\Big(e^{\sigma(gh)}\Lambda(gh) - e^{\sigma(h)}(L_{\mathcal{X}})^h_*(\Lambda(h)) \\ -e^{\sigma(gh)}(R_{\mathcal{Y}})^g_*(\Lambda(g)) + e^{\sigma(h)}(R_{\mathcal{Y}} \circ L_{\mathcal{X}})^{\epsilon(x)}_*(\Lambda(\epsilon(x)))\Big)(\xi, \eta) = 0.$$

This implies that (5.49) holds. QED

Motivated by the above result, we introduce the following definition.

Definition 5.16 *Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. A multivector field P on G is σ-*affine *if for any $g, h \in G$ such that $\alpha(g) = \beta(h) = x$ and any (local) bisections $\mathcal{X}, \mathcal{Y}$ through the points g, h, $\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$, we have*

$$\begin{aligned} P(gh) &= (R_{\mathcal{Y}})^g_*(P(g)) + e^{-\sigma(g)}(L_{\mathcal{X}})^h_*(P(h)) \\ &\quad -e^{-\sigma(g)}(L_{\mathcal{X}} \circ R_{\mathcal{Y}})^{\epsilon(x)}_*(P(\epsilon(x))). \end{aligned} \tag{5.50}$$

It is clear that if P is a σ-affine multivector and σ identically vanishes, then P is affine (see [85, 120]).
The following proposition gives a very useful characterization of σ-affine multivector fields.

Proposition 5.17 *Let $G \rightrightarrows M$ be an α-connected Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function on G. For a multivector field P on G, the following statements are equivalent:*

i) P is σ-affine;

ii) For any left-invariant vector field $\overleftarrow{X}$, the Lie derivative $e^\sigma(\mathcal{L}_0)_{\overleftarrow{X}}P$ is left-invariant.

Proof: The result follows using the fact that σ is multiplicative and proceeding as in the proof of Theorem 2.2 in [85]. QED

5.3 Examples of Jacobi groupoids

5.3.1 Poisson groupoids

If $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid with $E = 0$ and $\sigma = 0$ then, using Remark 5.8 and Theorem 5.13, we deduce that $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism from the cotangent groupoid $T^*G \rightrightarrows A^*G$ to the tangent groupoid $TG \rightrightarrows TM$. Thus, we recover the definition of a Poisson groupoid (see [83, 85]).

5.3.2 Contact groupoids

Let $(G \rightrightarrows M, \eta, \sigma)$ be a contact groupoid. If (Λ, E) is the Jacobi structure associated with the contact 1-form η then, using the results in Section 5.1, we have that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.

5.3.3 Locally conformal symplectic groupoids

In this Section, we will study a Jacobi groupoid $(G \rightrightarrows M, \Lambda, E, \sigma)$ such that its Jacobi structure (Λ, E) is l.c.s.. For this purpose, we introduce the following definition.

Definition 5.18 *Let $G \rightrightarrows M$ be a Lie groupoid with structural functions α, β, m and ϵ, (Ω, ω) be a l.c.s. structure on G, $\sigma : G \to \mathbb{R}$ be a multiplicative function and θ be the 1-form on G defined by*

$$\theta = e^{\sigma}(d_0\sigma - \omega). \tag{5.51}$$

Then, $(G \rightrightarrows M, \Omega, \omega, \sigma)$ is a locally conformal symplectic groupoid *(l.c.s. groupoid) if the following properties hold:*

$$m^*\Omega = \tau_1^*\Omega + e^{(\sigma\circ\tau_1)}\tau_2^*\Omega; \tag{5.52}$$

$$\tilde{\alpha} \circ \omega = 0, \quad \tilde{\beta} \circ \theta = 0; \tag{5.53}$$

$$m^*\omega = \tau_1^*\omega, \quad m^*\theta = e^{(\sigma\circ\tau_1)}\tau_2^*\theta; \tag{5.54}$$

$$\Lambda(\omega,\theta)=0, \quad (\theta+\omega-\tilde{\epsilon}\circ\tilde{\beta}\circ\omega)\circ\epsilon=0; \tag{5.55}$$

where $\tau_i : G^{(2)} \to G$, $i=1,2$, are the canonical projections, (Λ, E) is the Jacobi structure associated with the l.c.s. structure (Ω,ω) and $\tilde{\alpha}$, $\tilde{\beta}$, $\oplus_{T^*G}$ and $\tilde{\epsilon}$ are the structural functions of the cotangent Lie groupoid $T^*G \rightrightarrows A^*G$.

Two examples of this situation are the following ones.

Examples 5.19 1.- Let $(G \rightrightarrows M, \Omega)$ be a symplectic groupoid. This condition is equivalent to say that Ω satisfies the condition $m^*\Omega = \tau_1^*\Omega + \tau_2^*\Omega$ (see [19]). Therefore, we conclude that $(G \rightrightarrows M, \Omega)$ is a symplectic groupoid if and only if $(G \rightrightarrows M, \Omega, 0, 0)$ is a l.c.s. groupoid.

2.- Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, using the multiplicative character of σ, we can define a right action of $G \rightrightarrows M$ on the canonical projection $\pi_1 : M \times \mathbb{R} \to M$ as follows

$$(x,t)\cdot g = (\alpha(g), \sigma(g)+t) \tag{5.56}$$

for $(x,t) \in M \times \mathbb{R}$ and $g \in G$ such that $\beta(g) = x$. Thus, we have the corresponding action groupoid $(M\times\mathbb{R}) * G \rightrightarrows M \times \mathbb{R}$ (see Example 4 in Section 1.3.2). Moreover, if $(AG, [\![\ ,\ ]\!], \rho)$ is the Lie algebroid of G, the multiplicative function σ induces a 1-cocycle ϕ_0 on AG given by

$$\phi_0(x)(X_x) = X_x(\sigma), \tag{5.57}$$

for $x \in M$ and $X_x \in A_xG$. In addition, using the results in Section 1.3.2 (see Example 6 in Section 1.2.2), we deduce that the $\mathbb{R}$-linear map $* : \Gamma(AG) \to \mathfrak{X}(M \times \mathbb{R})$ defined by

$$X^* = (\rho(X)\circ\pi_1) + (\phi_0(X)\circ\pi_1)\frac{\partial}{\partial t} \tag{5.58}$$

induces an action of AG on the projection $\pi_1 : M\times\mathbb{R} \to M$ and the Lie algebroid of $(M\times\mathbb{R}) * G$ is just the action Lie algebroid $AG \ltimes \pi_1$.

Now, it is easy to prove that $(M\times\mathbb{R}) * G$ may be identified with the product manifold $G \times \mathbb{R}$ and, under this identification, the structural functions of the

Lie groupoid are given by

$$\begin{array}{rcl} \alpha_\sigma(g,t) & = & (\alpha(g),\sigma(g)+t),\ \text{for } (g,t)\in G\times\mathbb{R}, \\ \beta_\sigma(h,s) & = & (\beta(h),s),\ \text{for } (h,s)\in G\times\mathbb{R}, \\ m_\sigma((g,t),(h,s)), & = & (gh,t),\ \text{if } \alpha_\sigma(g,t)=\beta_\sigma(h,s), \\ \epsilon_\sigma(x,t) & = & (\epsilon(x),t),\ \text{for } (x,t)\in M\times\mathbb{R}. \end{array} \tag{5.59}$$

Thus, if $A(G\times\mathbb{R})$ is the Lie algebroid of $G\times\mathbb{R}$ and $X\in A_{(x,t)}(G\times\mathbb{R})$, it is clear that $X\in A_xG$ and therefore the map $\mathcal{J}: A(G\times\mathbb{R})\to AG\times\mathbb{R}$ defined by

$$X\in A_{(x,t)}(G\times\mathbb{R})\to\mathcal{J}(X)=(X,t)\in A_xG\times\mathbb{R} \tag{5.60}$$

defines an isomorphism of vector bundles. Furthermore, if on $AG\times\mathbb{R}$ we consider the Lie algebroid structure $(\llbracket\ ,\ \rrbracket^{\bar{}\phi_0},\bar\rho^{\phi_0})$ given by (3.75) then $\mathcal{J}$ is a Lie algebroid isomorphism. In conclusion, the Lie algebroid of the Lie groupoid $G\times\mathbb{R}\rightrightarrows M\times\mathbb{R}$ may be identified with $(AG\times\mathbb{R},\llbracket\ ,\ \rrbracket^{\bar{}\phi_0},\bar\rho^{\phi_0})$.

From (1.55) and (5.59), it follows that the projections $(\alpha_\sigma)^T$, $(\beta_\sigma)^T$, the inclusion $(\epsilon_\sigma)^T$ and the partial multiplication of the tangent Lie groupoid $T(G\times M)\rightrightarrows T(M\times\mathbb{R})$ are given by

$$\begin{array}{l} (\alpha_\sigma)^T(X_g+\lambda\frac{\partial}{\partial t}_{|t}) \\ \quad =\alpha^T(X_g)+(\lambda+X_g(\sigma))\frac{\partial}{\partial t}_{|t+\sigma(g)},\ \text{for } X_g+\lambda\frac{\partial}{\partial t}_{|t}\in T_{(g,t)}(G\times\mathbb{R}), \\ (\beta_\sigma)^T(Y_h+\mu\frac{\partial}{\partial t}_{|s}) \\ \quad =\beta^T(Y_h)+\mu\frac{\partial}{\partial t}_{|s},\ \text{for } Y_h+\mu\frac{\partial}{\partial t}_{|s}\in T_{(h,s)}(G\times\mathbb{R}), \\ (X_g+\lambda\frac{\partial}{\partial t}_{|t})\oplus_{T(G\times\mathbb{R})}(Y_h+\mu\frac{\partial}{\partial t}_{|s})=(X_g\oplus_{TG}Y_h)+\lambda\frac{\partial}{\partial t}_{|t}, \\ (\epsilon_\sigma)^T(X_x+\lambda\frac{\partial}{\partial t}_{|t})=\epsilon^T(X_x)+\lambda\frac{\partial}{\partial t}_{|t},\ \text{for } X_x+\lambda\dfrac{\partial}{\partial t}_{|t}\in T_{(x,t)}(M\times\mathbb{R}). \end{array} \tag{5.61}$$

On the other hand, using (1.58) and (5.59), we deduce that the projections $\widetilde{\alpha_\sigma},\widetilde{\beta_\sigma}$, the inclusion $\widetilde{\epsilon_\sigma}$ and the partial multiplication $\oplus_{T^*(G\times\mathbb{R})}$ in the cotangent groupoid $T^*(G\times\mathbb{R}) \rightrightarrows A^*G\times\mathbb{R}$ are defined by

$$
\begin{aligned}
&\widetilde{\alpha_\sigma}(\mu_g+\gamma d_0t_{|t})\\
&\quad=(\tilde{\alpha}(\mu_g),\sigma(g)+t),\ \text{ for } \mu_g+\gamma d_0t_{|t}\in T^*_{(g,t)}(G\times\mathbb{R}),\\
&\widetilde{\beta_\sigma}(\nu_h+\zeta d_0t_{|s})\\
&\quad=(\tilde{\beta}(\nu_h)-\zeta(d_0\sigma)(\epsilon(\beta(g))),s),\ \text{ for } \nu_h+\zeta d_0t_{|s}\in T^*_{(h,s)}(G\times\mathbb{R}),\\
&(\mu_g+\gamma d_0t_{|t})\oplus_{T^*(G\times\mathbb{R})}(\nu_h+\zeta d_0t_{|s})\\
&\qquad\qquad=(\mu_g+\zeta(d_0\sigma)(g))\oplus_{T^*G}\nu_h+(\gamma+\zeta)d_0t_{|t}\\
&\widetilde{\epsilon_\sigma}(\mu_x,t)=\tilde{\epsilon}(\mu_x)+0\,d_0t_{|t},\ \text{ for } (\mu_x,t)\in A^*_xG\times\mathbb{R}
\end{aligned}
\tag{5.62}
$$

Now, suppose that η is a contact 1-form on G in such a way that $(G \rightrightarrows M,\eta,\sigma)$ is a contact groupoid. If $\bar{\pi}_1 : G\times\mathbb{R}\to G$ and $\bar{\pi}_2 : G\times\mathbb{R}\to\mathbb{R}$ are the canonical projections, then the function $\bar{\sigma}$ on $G\times\mathbb{R}$ defined by $\bar{\sigma}=\sigma\circ\bar{\pi}_1$ is multiplicative and the 2-form Ω on $G\times\mathbb{R}$ given by

$$
\Omega=-(\bar{\pi}_1^*(d_0\eta)+\bar{\pi}_2^*(d_0t)\wedge\bar{\pi}_1^*(\eta)),
\tag{5.63}
$$

is a l.c.s. structure on $G\times\mathbb{R}$ with Lee 1-form $\omega=-\bar{\pi}_2^*(d_0t)$.

Next, we will prove that $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\Omega,\omega,\bar{\sigma})$ is a l.c.s. groupoid.

In fact, from (5.1), (5.3), (5.61) and (5.63), we deduce that

$$
m_\sigma^*\Omega=\tau_1^*\Omega+e^{(\bar{\sigma}\circ\tau_1)}\tau_2^*\Omega,
$$

$\tau_i : (G\times\mathbb{R})^{(2)}\to G\times\mathbb{R}$ being the canonical projections. Moreover, using (5.17) and (5.62), we obtain that

$$
\widetilde{\alpha_\sigma}\circ\omega=0,\qquad \widetilde{\beta_\sigma}\circ\theta=0,
$$

where θ is the 1-form on $G\times\mathbb{R}$ defined by $\theta=e^{\bar{\sigma}}(d_0\bar{\sigma}-\omega)$.

In addition, from (5.17), (5.59) and (5.61), it follows that

$$
m_\sigma^*\omega=\tau_1^*\omega,\qquad m_\sigma^*\theta=e^{(\bar{\sigma}\circ\tau_1)}\tau_2^*\theta.
$$

Furthermore, if (Λ,E) is the Jacobi structure on $G\times\mathbb{R}$ associated with the l.c.s. structure Ω and ξ is the Reeb vector field on G of the contact structure

η then, using (1.9), (1.12) and (5.63), we have that $E = -\xi$. Thus, from Remark 1.1 and Proposition 5.3, we conclude that

$$\Lambda(\omega, \theta) = \theta(\#_\Lambda(\omega)) = \theta(\xi) = e^{\bar{\sigma}}(\xi(\sigma) - \omega(\xi)) = 0.$$

Finally, a direct computation, using (5.17) and (5.62), proves that

$$(\theta + \omega - \widetilde{\epsilon_\sigma} \circ \widetilde{\beta_\sigma} \circ \omega) \circ \epsilon_\sigma = 0.$$

Now, we will show that a l.c.s. symplectic groupoid is a Jacobi groupoid $(G \rightrightarrows M, \Lambda, E, \sigma)$ such that (Λ, E) is the Jacobi structure associated with a l.c.s. structure.

Theorem 5.20 *Let $G \rightrightarrows M$ be a Lie groupoid, (Ω, ω) be a l.c.s. structure on G and $\sigma : G \to \mathbb{R}$ be a multiplicative function. If (Λ, E) is the Jacobi structure associated with the l.c.s. structure (Ω, ω) then $(G \rightrightarrows M, \Omega, \omega, \sigma)$ is a l.c.s. groupoid if and only if $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid.*

Proof: Assume that $(G \rightrightarrows M, \Omega, \omega, \sigma)$ is a l.c.s. groupoid and denote by $\tilde{\alpha}^*_\sigma$, $\tilde{\beta}^*_\sigma$, $\oplus^\sigma_{T^*G}$ and $\tilde{\epsilon}^*_\sigma$ the structural functions of the σ-cotangent groupoid $T^*G \rightrightarrows A^*G$. Using (5.52), we obtain that

$$\Omega(\epsilon(x))(\epsilon^x_*(X_x), \epsilon^x_*(Y_x)) = 0, \text{for } X_x, Y_x \in T_xM. \tag{5.64}$$

Now, suppose that $g \in G$ and that $X_g \in T_gG$ and $Z_{\alpha(g)} \in A_{\alpha(g)}G$. Then, from (1.55), it follows that

$$\begin{aligned} X_g &= X_g \oplus_{TG} \epsilon^{\alpha(g)}_*(\alpha^g_*(X_g)), \\ (L_g)^{\epsilon(\alpha(g))}_*(Z_{\alpha(g)}) &= 0_g \oplus_{TG} Z_{\alpha(g)}. \end{aligned}$$

Using these facts, (1.55), (1.58), (5.23), (5.52) and (5.64), we deduce that

$$i_{(\epsilon^{\alpha(g)}_*(\alpha^g_*(X_g)))}\Omega(\epsilon(\alpha(g))) = \tilde{\epsilon}(\tilde{\alpha}^*_\sigma(i_{X_g}\Omega(g))). \tag{5.65}$$

In a similar way, we prove that

$$i_{(\epsilon^{\beta(h)}_*(\beta^h_*(Y_h)))}\Omega(\epsilon(\beta(h))) = \tilde{\epsilon}(\tilde{\beta}^*_\sigma(i_{Y_h}\Omega(h))), \tag{5.66}$$

for $Y_h \in T_hG$.

Thus, if $\mu_g \in T_g^*G$ and $\nu_h \in T_h^*G$ satisfy $\tilde{\alpha}_\sigma^*(\mu_g) = \tilde{\beta}_\sigma^*(\nu_h)$ then, from (5.65) and (5.66) (taking $X_g = \#_\Lambda(\mu_g)$ and $Y_h = \#_\Lambda(\nu_h)$), we have that

$$i_{(\epsilon_*^{\alpha(g)}(\alpha_*^g(\#_\Lambda(\mu_g))))}\Omega(\epsilon(\alpha(g))) = i_{(\epsilon_*^{\beta(h)}(\beta_*^h(\#_\Lambda(\nu_h))))}\Omega(\epsilon(\beta(h))),$$

which implies that $\epsilon^T(\alpha^T(\#_\Lambda(\mu_g)) = \epsilon^T(\beta^T(\#_\Lambda(\nu_h))$ and therefore

$$\alpha^T(\#_\Lambda(\mu_g)) = \beta^T(\#_\Lambda(\nu_h)).$$

Moreover, using (1.58), (5.23), (5.52) and Remark 1.1, we conclude that

$$\begin{aligned}(i_{\#_\Lambda(\mu_g\oplus^\sigma_{T^*G}\nu_h)}\Omega(gh))(X_g \oplus_{TG} Y_h)\\ = \ (i_{(\#_\Lambda(\mu_g)\oplus_{TG}\#_\Lambda(\nu_h))}\Omega(gh))(X_g \oplus_{TG} Y_h),\end{aligned}$$

for $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$. Consequently (see Remark 1.1), it follows that $\#_\Lambda(\mu_g \oplus^\sigma_{T^*G} \nu_h) = \#_\Lambda(\mu_g) \oplus_{TG} \#_\Lambda(\nu_h)$ and, thus, the map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism over some map $\tilde{\varphi}_0 : A^*G \to TM$, between the σ-cotangent groupoid $T^*G \rightrightarrows A^*G$ and the tangent groupoid $TG \rightrightarrows TM$. In particular, this implies that

$$\alpha^T \circ \#_\Lambda = \tilde{\varphi}_0 \circ \tilde{\alpha}_\sigma^*, \qquad \beta^T \circ \#_\Lambda = \tilde{\varphi}_0 \circ \tilde{\beta}_\sigma^*. \tag{5.67}$$

Using (5.53), (5.67) and since $E = -\#_\Lambda(\omega)$, we deduce that the vector field E is α-vertical.

Next, suppose that $(g, h) \in G^{(2)}$ and denote by $R_h : G_{\beta(h)} \to G_{\alpha(h)}$ the right-translation by h. Then, (1.12), (5.52) and (5.54) imply that

$$(i_{E(gh)}\Omega(gh))(X_g \oplus_{TG} Y_h) = (i_{((R_h)_*^g(E(g)))}\Omega(gh))(X_g \oplus_{TG} Y_h),$$

for $(X_g, Y_h) \in T_{(g,h)}G^{(2)}$. Consequently, E is a right-invariant vector field and there exists $X_0 \in \Gamma(AG)$ such that $E = -\overrightarrow{X}_0$.

On the other hand, if $\mathcal{H}^{(\Lambda,E)}_{e^\sigma}$ is the hamiltonian vector field of the function e^σ, it is clear that $\mathcal{H}^{(\Lambda,E)}_{e^\sigma} = \#_\Lambda(\theta)$. Using this equality, (5.53), (5.54), (5.67) and proceeding as in the proof of the fact that E is right-invariant, we conclude that $\mathcal{H}^{(\Lambda,E)}_{e^\sigma}$ is a left-invariant vector field. Furthermore, if x is a point of M, then relation (5.55) implies that $\mathcal{H}^{(\Lambda,E)}_{e^\sigma}(\epsilon(x)) = -\overleftarrow{X}_0(\epsilon(x))$. Thus, $\#_\Lambda(d_0\sigma) = \overrightarrow{X}_0 - e^{-\sigma}\overleftarrow{X}_0$.

Finally, since $\Lambda(\omega,\theta) = 0$ and $E = -\#_\Lambda(\omega)$, we obtain that $E(\sigma) = 0$. Therefore, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid (see Theorem 5.13).
In a similar way, one can prove the converse. $\boxed{QED}$

Remark 5.21 Using Theorem 5.20 (see also Section 5.3.1 and Examples 5.19) we directly deduce that a symplectic groupoid is a Poisson groupoid. This result was first proved in [117].

5.3.4 Jacobi-Lie groups

Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid over a single point, that is, G is a Lie group and $M = \{\mathfrak{e}\}$, $\mathfrak{e}$ being the identity element in G. Then, using Theorem 5.13 and Proposition 5.15, it follows that :

i) Λ is σ-multiplicative, i.e.,

$$\Lambda(gh) = (R_h)^g_*(\Lambda(g)) + e^{-\sigma(g)}(L_g)^h_*(\Lambda(h)), \tag{5.68}$$

for $g, h \in G$.

ii) E is a right-invariant vector field, $E(\mathfrak{e}) = -X_0$.

iii) $\#_\Lambda(d_0\sigma) = \overrightarrow{X}_0 - e^{-\sigma}\overleftarrow{X}_0$.

Conversely, suppose that G is a Lie group endowed with a Jacobi structure (Λ, E) and $\sigma : G \to \mathbb{R}$ is a multiplicative function in such a way that *i)*, *ii)* and *iii)* hold. If $\mu_g \in T^*_gG$ and $\nu_h \in T^*_hG$ satisfy $\tilde{\alpha}^*_\sigma(\mu_g) = \tilde{\beta}^*_\sigma(\nu_h)$ then, from (5.23), we have that

$$\tilde{\alpha}(\mu_g) = \tilde{\beta}(e^{\sigma(g)}\nu_h). \tag{5.69}$$

Thus, using (1.57), (1.58), (1.59), (5.23), (5.68) and (5.69), we deduce that

$$\#_\Lambda(\mu_g \oplus^\sigma_{T^*G} \nu_h) = \#_\Lambda(\mu_g) \oplus_{TG} \#_\Lambda(\nu_h).$$

Therefore, the map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism from the σ-cotangent groupoid to the tangent groupoid.

On the other hand, from *ii)* and since σ is a multiplicative funcion, we obtain that $X_0(\sigma) = 0$, which implies that $E(\sigma) = 0$. Consequently, we have proved that $(G \rightrightarrows \{\mathfrak{e}\}, \Lambda, E, \sigma)$ is a Jacobi groupoid (see Theorem 5.13).

If G is a Lie group, (Λ, E) is a Jacobi structure on G and $\sigma : G \to \mathbb{R}$ is a multiplicative function such that *i)*, *ii)* and *iii)* hold then (G, Λ, E, σ) is said to be a *Jacobi-Lie group*.

5.3.5 An abelian Jacobi groupoid

Let $(L, [\![\, , \,]\!], \rho)$ be a Lie algebroid over M and Π_{L^*} be the corresponding linear Poisson structure on the dual bundle L^* (see Example 8 in Section 1.2.2). We may consider on L^* the Lie groupoid structure for which $\alpha = \beta$ is the vector bundle projection and the partial multiplication is the addition in the fibers. Then, L^* with the Poisson structure Π_{L^*} is a Poisson groupoid (see [117]).

Now, suppose that $\mu_0 \in \Gamma(L^*)$ is a 1-cocycle of L and denote by $(\Lambda_{(L^*,\mu_0)}, E_{(L^*,\mu_0)})$ the Jacobi structure on L^* given by (2.7). Note that:

i) The Liouville vector field Δ_{L^*} of L^* and the vertical lift $\mu_0^{\mathbf{v}} \in \mathfrak{X}(L^*)$ of μ_0 to L^* are α-vertical and β-vertical vector fields on L^*, and

ii) $\mu_0^{\mathbf{v}}$ is a right-invariant and left-invariant vector field on L^*.

Using *i)*, *ii)*, (1.56), (2.7), (5.8), (5.15) and the fact that (L^*, Π_{L^*}) is a Poisson groupoid, we deduce that $(L^* \rightrightarrows M, \Lambda_{(L^*,\mu_0)}, E_{(L^*,\mu_0)}, 0)$ is a Jacobi groupoid.

5.3.6 The banal Jacobi groupoid

Let M be a differentiable manifold. The results in Section 1.3.2 (see Examples 2 and 3 in Section 1.3.2) imply that $G = M \times \mathbb{R} \times M$ is a Lie groupoid over M and, moreover, the function $\sigma : G \to \mathbb{R}$ given by $\sigma(x, t, y) = t$ is multiplicative. Thus, we can consider the corresponding Lie groupoids

$TG \times \mathbb{R} \rightrightarrows TM \times \mathbb{R}$ and $T^*G \times \mathbb{R} \rightrightarrows A^*G$ with structural functions defined by (5.8) and (5.15).

On the other hand, the map $\Phi : TM \times \mathbb{R} \to AG$ given by

$$\Phi(X_x, \lambda) = (0, \lambda \frac{\partial}{\partial t}_{|0}, X_x) \in T_{(x,0,x)}G, \tag{5.70}$$

for $(X_x, \lambda) \in T_xM \times \mathbb{R}$, defines an isomorphism between the Lie algebroids $(TM \times \mathbb{R}, [\ ,\ ], \pi)$ (see Section 1.2.2) and AG. Thus, AG may be identified with $TM \times \mathbb{R}$ and, under this identification, the projections and the partial multiplications on $TG \times \mathbb{R}$ and $T^*G \times \mathbb{R}$ are given by

$$\begin{aligned}
&(\alpha^T)_\sigma((X_x, a\tfrac{\partial}{\partial t}_{|t}, Y_y), \lambda) = (Y_y, a + \lambda),\\
&(\beta^T)_\sigma((X'_{x'}, a'\tfrac{\partial}{\partial t}_{|t'}, Y'_{y'}), \lambda') = (X'_{x'}, \lambda'),\\
&((X_x, a\tfrac{\partial}{\partial t}_{|t}, Y_y), \lambda) \oplus_{TG\times\mathbb{R}} ((Y_y, a'\tfrac{\partial}{\partial t}_{|t'}, Y'_{y'}), a + \lambda)\\
&\qquad\qquad\qquad\qquad = ((X_x, (a + a')\tfrac{\partial}{\partial t}_{|t+t'}, Y'_{y'}), \lambda),
\end{aligned}$$

$$\begin{aligned}
&\tilde{\alpha}_\sigma((\mu_x, a\, d_0t_{|t}, \theta_y), \gamma) = (e^{-t}\theta_y, \gamma),\\
&\tilde{\beta}_\sigma((\mu'_{x'}, a'\, d_0t_{|t'}, \theta'_{y'}), \gamma') = (-\mu'_{x'}, a' - \gamma'),\\
&((\mu_x, a\, d_0t_{|t}, \theta_y), \gamma) \oplus_{T^*G\times\mathbb{R}} ((-e^{-t}\theta_y, a'\, d_0t_{|t'}, \theta'_{y'}), a' - e^{-t}a)\\
&\qquad\qquad\qquad\qquad = ((\mu_x, a'e^t\, d_0t_{|t+t'}, e^t\theta'_{y'}), \gamma - a + e^t\, a').
\end{aligned}$$

Now, suppose that (Λ, E) is a Jacobi structure on M. Then, it was proved in [43] that the pair (Λ', E') is a Jacobi structure on G, where

$$\begin{aligned}
\Lambda'(x, t, y) &= -\Big(\Lambda(x) - \tfrac{\partial}{\partial t}_{|t} \wedge E(x)\Big) + e^{-t}\Big(\Lambda(y) + \tfrac{\partial}{\partial t}_{|t} \wedge E(y)\Big),\\
E'(x, t, y) &= -E(x).
\end{aligned} \tag{5.71}$$

Furthermore, it follows that the map $\varphi_0 : A^*G \cong T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ given by (5.33) is just the homomorphism $\#_{(\Lambda,E)} : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$. Using the above facts, we conclude that $(G \rightrightarrows M, \Lambda', E', \sigma)$ is a Jacobi groupoid.

5.4 Jacobi groupoids and Jacobi bialgebroids

The aim of this Section is to show the relation between Jacobi groupoids and Jacobi bialgebroids.

5.4.1 Coisotropic submanifolds of a Jacobi manifold and Jacobi algebroids

In this Section, we will prove that if S is a coisotropic submanifold of a Jacobi manifold M then there exists a Jacobi algebroid structure on the conormal bundle to S. For this purpose, we will need the following result.

Lemma 5.22 *Let (M, Λ, E) be a Jacobi manifold and $([\![\, ,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ be the Lie algebroid structure on $T^*M \times \mathbb{R}$. Suppose that S is a coisotropic submanifold of M and that $\bar{\jmath}^* : \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \Omega^1(S) \times C^\infty(S,\mathbb{R})$ is the map defined by $\bar{\jmath}^*(\mu, f) = (\jmath^*\mu, \jmath^* f)$, $\jmath : S \to M$ being the canonical inclusion. Then:*

- *i) $Ker\, \bar{\jmath}^*$ is a Lie subalgebra of the Lie algebra $(\Omega^1(M) \times C^\infty(M,\mathbb{R}), [\![\, ,\,]\!]_{(\Lambda,E)})$.*
- *ii) The subspace of $\Omega^1(M) \times C^\infty(M,\mathbb{R})$ defined by $\{(\mu, f) \in \Omega^1(M) \times C^\infty(M,\mathbb{R}) \,/\, \mu_{|S} = 0, \jmath^* f = 0\}$ is an ideal in $Ker\, \bar{\jmath}^*$.*

Proof: i) If $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ satisfy

$$\bar{\jmath}^*(\mu, f) = 0, \quad \bar{\jmath}^*(\nu, g) = 0,$$

it follows from (1.29) that

$$\begin{aligned} &\bar{\jmath}^*[\![(\mu, f), (\nu, g)]\!]_{(\Lambda,E)} \\ &\qquad = (\jmath^*(i_{\#_\Lambda(\mu)} d_0\nu - i_{\#_\Lambda(\nu)} d_0\mu - d_0(\mu(\#_\Lambda(\nu)))), \\ &\qquad\quad \jmath^*(\mu(\#_\Lambda(\nu)) + \#_\Lambda(\mu)(g) - \#_\Lambda(\nu)(f))). \end{aligned} \tag{5.72}$$

Now, since $\jmath^*\mu = 0, \jmath^*\nu = 0$ and S is a coisotropic submanifold, it follows that the restriction to S of the vector fields $\#_\Lambda(\mu)$ and $\#_\Lambda(\nu)$ is tangent to S. Thus, from (5.72), we deduce that

$$\bar{\jmath}^*[\![(\mu, f), (\nu, g)]\!]_{(\Lambda,E)} = 0.$$

ii) If μ' and ν' are 1-form on M, we will denote by $[\![\mu', \nu']\!]_\Lambda$ the 1-form on M given by

$$[\![\mu', \nu']\!]_\Lambda = i_{\#_\Lambda(\mu')} d_0\nu' - i_{\#_\Lambda(\nu')} d_0\mu' - d_0(\mu'(\#_\Lambda(\nu'))).$$

Note that

$$[\![\mu', f\nu']\!]_\Lambda = f[\![\mu', \nu']\!]_\Lambda + \#_\Lambda(\mu')(f)\nu', \tag{5.73}$$

for $f \in C^\infty(M, \mathbb{R})$.

Next, suppose that $(\mu, f), (\nu, g) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ satisfy the following conditions

$$\mu_{|S} = 0, \quad \jmath^* f = 0, \quad \bar{\jmath}^*(\nu, g) = 0.$$

Then, proceeding as in the proof of *i)*, we have that

$$[\![(\mu, f), (\nu, g)]\!]_{(\Lambda,E)|S} = ([\![\mu, \nu]\!]_{\Lambda|S}, 0).$$

Thus, if x is a point of S, we must prove that $[\![\mu, \nu]\!]_\Lambda(x) = 0$. For this purpose, we consider a coordinate neighborhood (U, φ) of M with coordinates $(x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ such that

$$\varphi(U \cap S) = \{(x_1, \dots, x_m) \in \varphi(U) \,/\, x_{n+1} = \ldots = x_m = 0\}.$$

Here, n (respectively, m) is the dimension of S (respectively, M). Then, on U

$$\mu = \sum_{i=1}^{m} \mu^i \, d_0x_i, \qquad \nu = \sum_{j=1}^{n} \nu^j \, d_0x_j + \sum_{k=n+1}^{m} \bar{\nu}^k \, d_0x_k, \tag{5.74}$$

with

$$\jmath^*\mu^i = 0, \quad \jmath^*\nu^j = 0, \tag{5.75}$$

for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Note that, since S is a coisotropic submanifold of M, it follows that

$$\#_\Lambda(d_0x_k)_{|S}(\mu^i) = 0, \tag{5.76}$$

for all $i \in \{1, \dots, m\}$ and $k \in \{n+1, \dots, m\}$. Therefore, using (5.73)-(5.76), we conclude that $[\![\mu, \nu]\!]_\Lambda(x) = 0$. QED

Now, we will show the main result of the Section.

Proposition 5.23 *Let (M, Λ, E) be a Jacobi manifold and $([\![\,,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ be the Lie algebroid structure on $T^*M \times \mathbb{R}$. Suppose that S is a coisotropic submanifold of M. Then:*

i) The conormal bundle to S*,* $N(S) = (TS)^\circ \to S$*, admits a Lie algebroid structure* $(\llbracket\, ,\, \rrbracket_S, \rho_S)$ *defined by*

$$\begin{array}{rcl} \llbracket \mu, \nu \rrbracket_S(x) & = & (\pi_1 \llbracket (\tilde{\mu}, 0), (\tilde{\nu}, 0) \rrbracket_{(\Lambda,E)})(x), \\ \rho_S(\mu)(x) & = & \#_\Lambda(\mu_x), \end{array} \tag{5.77}$$

for $\mu, \nu \in \Gamma(TS^\circ)$ *and* $x \in S$*, where* $\pi_1 : \Omega^1(M) \times C^\infty(M, \mathbb{R}) \to \Omega^1(M)$ *is the projection onto the first factor and* $\tilde{\mu}$ *and* $\tilde{\nu}$ *are arbitrary extensions to* M *of* μ *and* ν*, respectively.*

ii) The section E_S *of the vector bundle* $N(S)^* \to S$ *characterized by*

$$\mu(E_S(x)) = -\mu(E(x)), \tag{5.78}$$

for all $\mu \in N_x S = (T_x S)^\circ$ *and* $x \in S$*, is a 1-cocycle of the Lie algebroid* $(N(S), \llbracket\, ,\, \rrbracket_S, \rho_S)$*.*

Proof: i) follows from Lemma 5.22 and *ii)* follows using (5.78) and the fact that $(-E, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle of the Lie algebroid $(T^*M \times \mathbb{R}, \llbracket\, ,\, \rrbracket_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ (see Example 4 in Section 1.2.2). QED

Remark 5.24 If the Jacobi manifold M is Poisson (that is, $E = 0$) then the 1-cocycle E_S identically vanishes and $(\llbracket\, ,\, \rrbracket_S, \rho_S)$ is just the Lie algebroid structure obtained by Weinstein in [117].

5.4.2 The Jacobi bialgebroid of a Jacobi groupoid

In this Section, we will show that Jacobi bialgebroids are the infinitesimal invariants for Jacobi groupoids.

Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid and $(AG, \llbracket\, ,\, \rrbracket, \rho)$ be the Lie algebroid of G. Then, E is a right-invariant vector field and, thus, there exists a section X_0 of AG such that $E = -\overrightarrow{X_0}$ (see Theorem 5.13). Moreover, the conormal bundle to M, as a submanifold of G, may be identified with A^*G. In fact, the inclusion $\tilde{\epsilon} : A^*G \to T^*G$ of the cotangent groupoid $T^*G \rightrightarrows A^*G$ induces an isomorphism between A^*G and the conormal bundle to M.

Now, we consider the section ϕ_0 of A^*G given by

$$\phi_0(X_x) = X_x(\sigma), \tag{5.79}$$

for $X_x \in A_xG$ and $x \in M$. Since σ is a Lie groupoid 1-cocycle, it follows that ϕ_0 is a 1-cocycle of the Lie algebroid AG (see [120]).

On the other hand, using that $M \cong \epsilon(M)$ is a coisotropic submanifold of G, we deduce that there exists a Lie algebroid structure $([\![\, , \,]\!]_*, \rho_*)$ on A^*G and, furthermore, the vector field E induces a 1-cocycle $E_M \in \Gamma(AG)$ of A^*G (see Proposition 5.23). In fact, from Proposition 5.23, we have that $E_M = X_0$ and

$$\begin{array}{rcl} [\![\mu, \nu]\!]_*(x) & = & \pi_1 [\![(\widetilde{\tilde{\epsilon} \circ \mu}, 0), (\widetilde{\tilde{\epsilon} \circ \nu}, 0)]\!]_{(\Lambda, E)}(\epsilon(x)), \\ \rho_*(\mu)(x) & = & \alpha_*^{\epsilon(x)}(\#_\Lambda(\tilde{\epsilon}(\mu_x))), \end{array} \tag{5.80}$$

for $\mu, \nu \in \Gamma(A^*G)$ and $x \in M$, where $\widetilde{\tilde{\epsilon} \circ \mu}$ and $\widetilde{\tilde{\epsilon} \circ \nu}$ are arbitrary extensions to G of $\tilde{\epsilon} \circ \mu$ and $\tilde{\epsilon} \circ \nu$, respectively. Note that, since $M \cong \epsilon(M)$ is a coisotropic submanifold of G (see Proposition 5.15), it follows that

$$\epsilon_*^x(\rho_*(\mu)(x)) = \#_\Lambda(\tilde{\epsilon}(\mu_x)). \tag{5.81}$$

Note that, from (5.33) and (5.80), we have that $\varphi_0 = (\rho_*, X_0)$, where

$$(\rho_*, X_0)(\mu_x) = (\rho_*(\mu_x), \mu_x(X_0(x))), \tag{5.82}$$

for $\mu_x \in A_x^*G$. In addition, we will prove the following result.

Theorem 5.25 *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid. Then, the pair $((AG, \phi_0), (A^*G, X_0))$ is a Jacobi bialgebroid.*

Proof: Denote by $d_*^{X_0}$ the X_0-differential of the Lie algebroid $(A^*G, [\![\, , \,]\!]_*, \rho_*)$. We will show that

$$e^\sigma (\mathcal{L}_0)_{\overleftarrow{X}} \Lambda = -\overleftarrow{d_*^{X_0} X}. \tag{5.83}$$

for $X \in \Gamma(AG)$. Suppose that μ_1, μ_2 are any sections of A^*G. Let $\widetilde{\tilde{\epsilon} \circ \mu_1}$, $\widetilde{\tilde{\epsilon} \circ \mu_2}$ be any of their extensions to 1-forms on G. Then, using (1.21), (1.29), (3.2), (5.80) and the fact that $\sigma_{|\epsilon(M)} \equiv 0$, we have that

$$\begin{aligned}
&\Big(e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{X}}\Lambda\Big)_{|\epsilon(M)}(\mu_1,\mu_2)\\
&\quad= \Big(((\mathcal{L}_0)_{\#_\Lambda(\widetilde{\tilde\epsilon\circ\mu_1})}\widetilde{\tilde\epsilon\circ\mu_2} - (\mathcal{L}_0)_{\#_\Lambda(\widetilde{\tilde\epsilon\circ\mu_2})}\widetilde{\tilde\epsilon\circ\mu_1} - \Lambda(\widetilde{\tilde\epsilon\circ\mu_1},\widetilde{\tilde\epsilon\circ\mu_2}))(\overleftarrow{X})\\
&\qquad +\#_\Lambda(\widetilde{\tilde\epsilon\circ\mu_2})(\widetilde{\tilde\epsilon\circ\mu_1}(\overleftarrow{X})) - \#_\Lambda(\widetilde{\tilde\epsilon\circ\mu_1})(\widetilde{\tilde\epsilon\circ\mu_2}(\overleftarrow{X}))\Big)_{|\epsilon(M)}\\
&\quad= [\![\mu_1,\mu_2]\!]_*(X) + \rho_*(\mu_2)(\mu_1(X))\\
&\qquad -\rho_*(\mu_1)(\mu_2(X)) - (X_0\wedge X)(\mu_1,\mu_2)\\
&\quad= -(d_*^{X_0}X)(\mu_1,\mu_2).
\end{aligned}$$

Thus, since $-\overleftarrow{d_*^{X_0}X}$ and $e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{X}}\Lambda$ are left-invariant 2-vectors (see Proposition 5.17) and their evaluation coincides on the conormal bundle A^*G, we deduce (5.83).

Using (1.53), (5.79) and (5.83), we obtain that

$$\begin{aligned}
\overleftarrow{d_*^{X_0}[\![X,Y]\!]} &= -e^{\sigma}(\mathcal{L}_0)_{[\overleftarrow{X},\overleftarrow{Y}]}\Lambda\\
&= (\mathcal{L}_0)_{\overleftarrow{Y}}(e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{X}}\Lambda) - \overleftarrow{Y}(\sigma)(e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{X}}\Lambda)\\
&\quad -(\mathcal{L}_0)_{\overleftarrow{X}}(e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{Y}}\Lambda) + \overleftarrow{X}(\sigma)(e^{\sigma}(\mathcal{L}_0)_{\overleftarrow{Y}}\Lambda)\\
&= \overleftarrow{[\![X,d_*^{X_0}Y]\!]} - \overleftarrow{\phi_0(X)d_*^{X_0}Y}\\
&\quad -\overleftarrow{[\![Y,d_*^{X_0}X]\!]} + \overleftarrow{\phi_0(Y)d_*^{X_0}X},
\end{aligned} \tag{5.84}$$

for $X,Y\in\Gamma(AG)$. Thus, from (3.22) and (5.84), we conclude that

$$d_*^{X_0}[\![X,Y]\!] = [\![X,d_*^{X_0}Y]\!]^{\phi_0} - [\![Y,d_*^{X_0}X]\!]^{\phi_0},$$

for $X,Y\in\Gamma(AG)$.

Now, (5.79), the condition $E(\sigma) = -\overrightarrow{X_0}(\sigma) = 0$ (see Theorem 5.13) and the fact that σ is a multiplicative function imply that $\phi_0(X_0)\circ\alpha = 0$ and, therefore,

$$\phi_0(X_0) = 0. \tag{5.85}$$

Furthermore, if $x\in M$ then, from (5.34), (5.79) and (5.81), we deduce that

$$\epsilon_*^x(\rho_*(\phi_0)(x)) = \#_\Lambda(d_0\sigma)(\epsilon(x)) = \overleftarrow{X_0}(\epsilon(x)) - \overrightarrow{X_0}(\epsilon(x)) = -\epsilon_*^x(\alpha_*^{\epsilon(x)}(X_0(x))),$$

that is, (see (1.52)),

$$\rho_*(\phi_0)(x) = -\rho(X_0)(x). \tag{5.86}$$

On the other hand, using (5.79), (5.83) and (5.85), it follows that

$$e^{-\sigma} i_{d_0\sigma}(\overleftarrow{d_* X}) = -i_{d_0\sigma}((\mathcal{L}_0)_{\overleftarrow{X}}\Lambda) + e^{-\sigma}(\phi_0(X)\circ\alpha)\overleftarrow{X_0}.$$

Consequently, using again (5.79), we have that

$$i_{\phi_0}(d_* X) = -(i_{d_0\sigma}((\mathcal{L}_0)_{\overleftarrow{X}}\Lambda))\circ\epsilon + \phi_0(X)X_0. \tag{5.87}$$

Finally, from (5.34) and (5.79), we deduce that

$$\begin{array}{rcl} 0 = [\overleftarrow{X}, \overrightarrow{X_0}] & = & i_{d_0\sigma}((\mathcal{L}_0)_{\overleftarrow{X}}\Lambda) + \#_\Lambda(d_0(\phi_0(X)\circ\alpha)) \\ & & -e^{-\sigma}(\phi_0(X)\circ\alpha)\overleftarrow{X_0} + e^{-\sigma}\overleftarrow{[\![X, X_0]\!]}, \end{array}$$

which implies that (see (1.21), (5.79), (5.80) and (5.87))

$$i_{\phi_0}(d_* X) + d_*(\phi_0(X)) + [\![X_0, X]\!] = 0.$$

QED

Before describing the examples, we will relate the Jacobi structure on G and the Jacobi structure on M induced by the Jacobi bialgebroid structure of Theorem 5.25.

Proposition 5.26 *Let $(G \rightrightarrows M, \Lambda, E, \sigma)$ be a Jacobi groupoid and (Λ_0, E_0) be the Jacobi structure on M induced by the Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$. Then, the projection β is a Jacobi antimorphism between the Jacobi manifolds (G, Λ, E) and (M, Λ_0, E_0) and the pair (α, e^σ) is a conformal Jacobi morphism.*

Proof: Denote by $\{\ ,\ \}_{(\Lambda,E)}$ (respectively, $\{\ ,\ \}_{(\Lambda_0,E_0)}$) the Jacobi bracket associated with the Jacobi structure (Λ, E) (respectively, (Λ_0, E_0)). Then, we must prove that

$$\{\beta^* f_1, \beta^* f_2\}_{(\Lambda,E)} = -\beta^*\{f_1, f_2\}_{(\Lambda_0,E_0)},$$
$$e^{-\sigma}\{e^\sigma\alpha^* f_1, e^\sigma\alpha^* f_2\}_{(\Lambda,E)} = \alpha^*\{f_1, f_2\}_{(\Lambda_0,E_0)},$$

for $f_1, f_2 \in C^\infty(M,\mathbb{R})$.

Now, if $(\rho_*, X_0) : \Gamma(A^*G) \to \mathfrak{X}(M) \times C^\infty(M,\mathbb{R})$ is the map given by (5.82) and $(\rho, \phi_0) : \Gamma(AG) \to \mathfrak{X}(M) \times C^\infty(M,\mathbb{R})$ is the homomorphism of $C^\infty(M,\mathbb{R})$-modules defined by (3.9) then, from (3.46), (3.47), (5.82), it follows that

$$\#_{(\Lambda_0,E_0)} = (\rho_*, X_0) \circ (\rho, \phi_0)^*, \tag{5.88}$$

where $(\rho, \phi_0)^* : \Omega^1(M) \times C^\infty(M,\mathbb{R}) \to \Gamma(A^*G)$ is the adjoint operator of the homomorphism (ρ, ϕ_0). In particular,

$$\#_{\Lambda_0}(\mu_0) = \rho_*(\rho^*(\mu_0)), \tag{5.89}$$

for $\mu_0 \in \Omega^1(M)$.

Using (5.8) and since $(\beta^T)_\sigma \circ \#_{(\Lambda,E)} = (\rho_*, X_0) \circ \tilde{\beta}_\sigma$ we have that

$$\begin{aligned} \{\beta^* f_1, \beta^* f_2\}_{(\Lambda,E)} &= \langle \#_{(\Lambda,E)}(\beta^* d_0 f_1, \beta^* f_1), (\beta^* d_0 f_2, \beta^* f_2)\rangle \\ &= \langle ((\beta)^T_\sigma \circ \#_{(\Lambda,E)})(\beta^* d_0 f_1, \beta^* f_1), (d_0 f_2 \circ \beta, \beta^* f_2)\rangle \\ &= \langle ((\rho_*, X_0) \circ \tilde{\beta}_\sigma)(\beta^* d_0 f_1, \beta^* f_1), (d_0 f_2 \circ \beta, \beta^* f_2)\rangle. \end{aligned}$$

From (1.52), (1.58), (3.9), (5.15) and (5.79), we deduce that

$$\tilde{\beta}_\sigma(((\beta_*^g)^*(\mu_{\beta(g)}), \lambda) = -(\rho, \phi_0)^*(\mu_{\beta(g)}, \lambda),$$

for $(\mu_{\beta(g)}, \lambda) \in T^*_{\beta(g)}M \times \mathbb{R}$. Using this fact and (5.88), we get that

$$\{\beta^* f_1, \beta^* f_2\}_{(\Lambda,E)} = \beta^*\{f_1, f_2\}_{(\Lambda_0,E_0)}$$

On the other hand, using (1.52), (5.8), (5.89), Remark 5.14 and since $(\alpha^T)_\sigma \circ \#_{(\Lambda,E)} = (\rho_*, X_0) \circ \tilde{\alpha}_\sigma$, we obtain that

$$\begin{aligned} &e^{-\sigma}\{e^\sigma \alpha^* f_1, e^\sigma \alpha^* f_2\}_{(\Lambda,E)} \\ &\quad = e^{-\sigma}\langle \#_{(\Lambda,E)}(d_0(e^\sigma \alpha^* f_1), e^\sigma \alpha^* f_1), (d_0(e^\sigma \alpha^* f_2), e^\sigma \alpha^* f_2)\rangle \\ &\quad = e^\sigma \langle ((\alpha^T)_\sigma \circ \#_{(\Lambda,E)})(\alpha^* d_0 f_1, \alpha^* f_1), (d_0 f_2 \circ \alpha, \alpha^* f_2)\rangle \\ &\qquad + e^\sigma (\alpha^* f_1)\langle \#_{(\Lambda,E)}(d_0\sigma, 0), (\alpha^*(d_0 f_2), \alpha^* f_2)\rangle \\ &\quad = e^\sigma \langle ((\rho_*, X_0) \circ \tilde{\alpha}_\sigma)(\alpha^*(d_0 f_1), \alpha^* f_1), (d_0 f_2 \circ \alpha, \alpha^* f_2)\rangle \\ &\qquad + \alpha^*(f_1 E_0(f_2)). \end{aligned}$$

Now, from (1.52), (1.58), (3.9), (5.15) and (5.79), it follows that

$$e^{\sigma(g)}\tilde{\alpha}_\sigma((\alpha_*^g)^*(\mu_{\alpha(g)}),\lambda) = (\rho,\phi_0)^*(\mu_{\alpha(g)},0),$$

for $(\mu_{\alpha(g)},\lambda) \in T^*_{\alpha(g)}M \times \mathbb{R}$. Therefore,

$$e^{-\sigma}\{e^\sigma\alpha^* f_1, e^\sigma\alpha^* f_2\}_{(\Lambda,E)} = \alpha^*\{f_1,f_2\}_{(\Lambda_0,E_0)}.$$

QED

Next, we will describe the Jacobi bialgebroids associated with some examples of Jacobi groupoids. We remark that two Jacobi bialgebroids $((A,\phi_0),(A^*,X_0))$ and $((B,\mu_0),(B^*,Y_0))$ over a manifold M are isomorphic if there exists a Lie algebroid isomorphism $\mathcal{I}: A \to B$ (over the identity $Id: M \to M$) such that $\mathcal{I}(X_0) = Y_0$ and, in addition, the adjoint operator $\mathcal{I}^*: B^* \to A^*$ is also a Lie algebroid isomorphism satisfying $\mathcal{I}^*(\mu_0) = \phi_0$.

Examples 5.27 1.-*Poisson groupoids*
If (G,Λ,E,σ) is a Jacobi groupoid with $E=0$ and $\sigma=0$, that is, (G,Λ) is a Poisson groupoid, then we have that ϕ_0 and X_0 identically vanish (see (5.79) and Remark 5.24). Therefore, (3.33) and Theorem 5.25 imply a well-known result (see [83, 120]): if (G,Λ) is a Poisson groupoid then the pair (AG, A^*G) is a Lie bialgebroid.

2.-*Contact groupoids*
Let $(G \rightrightarrows M,\eta,\sigma)$ be a contact groupoid and (Λ,E) be the Jacobi structure associated with the contact 1-form η. Then, $(G \rightrightarrows M,\Lambda,E,\sigma)$ is a Jacobi groupoid.
Now, denote by (Λ_0,E_0) the Jacobi structure on M characterized by the conditions (5.6), by X_0 the section of the Lie algebroid AG of G satisfying $E = -\overrightarrow{X_0}$ and by $\mathcal{I}: T^*M \times \mathbb{R} \to AG$ the Lie algebroid isomorphism given by (5.7). If we consider the section $(0,-1) \in \Omega^1(M) \times C^\infty(M,\mathbb{R})$ of the vector bundle $T^*M \times \mathbb{R} \to M$, we have that (see (5.7))

$$\mathcal{I}(0,-1) = X_0. \tag{5.90}$$

Moreover, if $\mathcal{I}^* : A^*G \to TM \times \mathbb{R}$ is the adjoint operator of $\mathcal{I}$, from (5.7), it follows that

$$\mathcal{I}^*(\nu_x) = (-\alpha_*^{\epsilon(x)}(\#_\Lambda(\tilde{\epsilon}(\nu_x))), -\nu_x(X_0(x))), \tag{5.91}$$

for $\nu_x \in A^*_xG$, where $\tilde{\epsilon}$ is the inclusion in the Lie groupoid $T^*G \rightrightarrows A^*G$.

Next, denote by $([\ ,\ ]_-, \pi_-)$ the Lie algebroid structure on the vector bundle $TM \times \mathbb{R} \to M$ defined by

$$[(X, f), (Y, g)]_- = (-[X, Y], -(X(g) - Y(f))), \qquad \pi_-(X, f) = -X,$$

for $(X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$.

On the other hand, if on the vector bundle $TG \times \mathbb{R} \to G$ we consider the natural Lie algebroid structure (see Section 1.2.2) then the map $\#_{(\Lambda,E)}$: $T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie algebroid homomorphism between the Lie algebroids $(T^*G\times\mathbb{R}, [\![\ ,\ ]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ and $TG\times\mathbb{R}$. Using this fact, (5.80) and since $M \cong \epsilon(M)$ is a coisotropic submanifold of G, we deduce that $\mathcal{I}^*$ defines an isomorphism between the Lie algebroids A^*G and $(TM\times\mathbb{R}, [\ ,\ ]_-, \pi_-)$. In addition, from (5.91) and Proposition 5.3, we obtain that $\mathcal{I}^*(\phi_0) = (-E_0, 0)$.

In conclusion, if on the vector bundle $T^*M \times \mathbb{R} \to M$ (respectively, $TM \times \mathbb{R} \to M$) we consider the Lie algebroid structure $([\![\ ,\ ]\!]_{(\Lambda_0,E_0)}, \widetilde{\#}_{(\Lambda_0,E_0)})$ (respectively, $([\ ,\ ]_-, \pi_-)$) then the Jacobi bialgebroids $((AG, \phi_0), (A^*G, X_0))$ and $((T^*M \times \mathbb{R}, (-E_0, 0)), (TM \times \mathbb{R}, (0, -1)))$ are isomorphic. Note that the Jacobi structure on M induced by the Jacobi bialgebroid $((T^*M \times \mathbb{R}, (-E_0, 0)), (TM \times \mathbb{R}, (0, -1)))$ is just (Λ_0, E_0) (see (3.45)).

3.-*L.c.s. groupoids*
Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid and θ be the 1-form on G given by (5.51). Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid, where (Λ, E) is the Jacobi structure associated with the l.c.s. structure (Ω, ω). Furthermore, the 1-form $e^{-\sigma}\theta$ is closed and since $\tilde{\beta} \circ \theta = 0$, it follows that θ is basic with respect to the projection α. Thus, there exists a unique closed 1-form θ_0 on M such that

$$\alpha^*\theta_0 = e^{-\sigma}\theta. \tag{5.92}$$

Note that if $Y_h \in T_hG$ then

$$\begin{array}{rcl}(\beta^*\theta_0)(h)(Y_h) &=& \theta_0(\beta(h))(\alpha_*^{\epsilon(\beta(h))}(\epsilon_*^{\beta(h)}(\beta_*^h(Y_h))))\\ &=& (e^{-\sigma}\theta)(\epsilon(\beta(h)))(\epsilon_*^{\beta(h)}(\beta_*^h(Y_h)))\end{array}$$

and, since σ is multiplicative, $i_E\Omega = \omega$ and E is a right-invariant vector field,

$$(\beta^*\theta_0)(h)(Y_h) = -\{(\epsilon\circ\beta)^*(\omega)\}(h)(Y_h) = -\omega(h)(Y_h),$$

that is,

$$\beta^*\theta_0 = -\omega. \tag{5.93}$$

On the other hand, $\#_\Lambda(\theta)$ is the hamiltonian vector field $\mathcal{H}_{e^\sigma}^{(\Lambda,E)}$ of the function e^σ. Moreover, from Theorems 3.13 and 5.20, and Proposition 5.26, we deduce that there exists a Jacobi structure (Λ_0, E_0) on M in such a way that the couple (α, e^σ) is a conformal Jacobi morphism between the Jacobi manifolds (G,Λ,E) and (M,Λ_0,E_0). This implies that

$$\begin{array}{rcl}\#_{\Lambda_0(\alpha(g))}(\theta_0(\alpha(g))) &=& e^{\sigma(g)}(\alpha_*^g\circ\#_{\Lambda(g)}\circ(\alpha_*^g)^*)(\theta_0(\alpha(g)))\\ &=& \alpha_*^g(\mathcal{H}_{e^\sigma}^{(\Lambda,E)}(g)) = E_0(\alpha(g)),\end{array}$$

for $g\in G$, where $(\alpha_*^g)^* : T^*_{\alpha(g)}M \to T^*_gG$ is the adjoint map of the tangent map $\alpha_*^g : T_gG \to T_{\alpha(g)}M$. Therefore, we have proved that $\#_{\Lambda_0}(\theta_0) = E_0$.

Next, we will describe the Lie algebroid associated with a l.c.s. groupoid.

Theorem 5.28 *Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid, AG be the Lie algebroid of G, (Λ, E) be the Jacobi structure on G associated with the l.c.s. structure (Ω,ω) and (Λ_0,E_0) be the corresponding Jacobi structure on M. Then, the map $\Psi : \Omega^1(M) \to \mathfrak{X}_L(G)$ between $\Omega^1(M)$ and the space $\mathfrak{X}_L(G)$ of left-invariant vector fields on G defined by $\Psi(\mu) = e^\sigma\#_\Lambda(\alpha^*\mu)$ induces an isomorphism between the vector bundles T^*M and AG. Under this isomorphism, the Lie bracket on $\Gamma(AG) \cong \mathfrak{X}_L(G)$ and the anchor map of AG are given by*

$$\begin{array}{rcl}[\![\mu,\nu]\!]_{(\Lambda_0,E_0,\theta_0)} &=& (\mathcal{L}_0)_{\#_{\Lambda_0}(\mu)}\nu - (\mathcal{L}_0)_{\#_{\Lambda_0}(\nu)}\mu - d_0(\Lambda_0(\mu,\nu))\\ && -i_{E_0}(\mu\wedge\nu) - \Lambda_0(\mu,\nu)\theta_0,\\ \widetilde{\#}_{(\Lambda_0,E_0,\theta_0)}(\mu) &=& \#_{\Lambda_0}(\mu),\end{array}$$

for $\mu,\nu\in\Omega^1(M)$, where θ_0 is the 1-form on M characterized by (5.92).

Proof: Let μ be a 1-form on M. Since the map $\#_\Lambda : T^*G \to TG$ is a morphism between the σ-cotangent groupoid and the tangent groupoid $TG \rightrightarrows TM$ over some map $\tilde{\varphi}_0 : A^*G \to TM$, we obtain the that vector field $\tilde{X} = \Psi(\mu)$ is β-vertical. In fact, if $g \in G$ then (see (1.58))

$$\begin{aligned} \beta^T(\tilde{X}(g)) &= e^{\sigma(g)}\beta^T(\#_\Lambda(\alpha^*\mu)(g)) \\ &= e^{\sigma(g)}\tilde{\varphi}_0(\tilde{\beta}((\alpha^*\mu)(g))) = 0. \end{aligned}$$

Moreover, if $(g,h) \in G^{(2)}$ and $L_g : G^{\alpha(g)} \to G^{\beta(g)}$ is the left-translation by g then, using Remark 1.1 and the fact that σ is multiplicative, we deduce that

$$(i_{\tilde{X}(gh)}\Omega(gh))(Y_g \oplus_{TG} Z_h) = -e^{\sigma(g)}e^{\sigma(h)}\mu(\alpha(h))(\alpha_*^h(Z_h)),$$

for $(Y_g, Z_h) \in T_{(g,h)}G^{(2)}$. Thus, from (5.52), it follows that

$$i_{\tilde{X}(gh)}\Omega(gh) = i_{0_g \oplus_{TG} \tilde{X}(h)}\Omega(gh)$$

and, as $\tilde{X}$ is a β-vertical vector field, we have that

$$i_{\tilde{X}(gh)}\Omega(gh) = i_{(L_g)_*^h(\tilde{X}(h))}\Omega(gh),$$

that is, $\tilde{X}(gh) = (L_g)_*^h(\tilde{X}(h))$. This proves that $\tilde{X} \in \mathfrak{X}_L(G)$.

Conversely, assume that $\tilde{X} \in \mathfrak{X}_L(G)$ and consider the 1-form on G defined by $\tilde{\mu} = -i_{\tilde{X}}\Omega$. We will show that $\Psi(\mu) = \tilde{X}$, where μ is the 1-form on M given by

$$\mu = \epsilon^*\tilde{\mu}.$$

If $g \in G$ and $Y_g \in T_gG$ then

$$\tilde{X}(g) = 0_g \oplus_{TG} \tilde{X}(\epsilon(\alpha(g))), \qquad Y_g = Y_g \oplus_{TG} \epsilon_*^{\alpha(g)}(\epsilon_*^g(Y_g)). \tag{5.94}$$

Therefore, using (5.52) and (5.94), we deduce that

$$\begin{aligned} \tilde{\mu}(g)(Y_g) &= e^{\sigma(g)}\Omega(\epsilon(\alpha(g)))(\tilde{X}(\epsilon(\alpha(g))), \epsilon_*^{\alpha(g)}(\epsilon_*^g(Y_g))) \\ &= e^{\sigma(g)}\tilde{\mu}(\epsilon(\alpha(g)))(\epsilon_*^{\alpha(g)}(\alpha_*^g(Y_g))) = e^{\sigma(g)}(\alpha^*\mu)(g)(Y_g). \end{aligned}$$

Consequently, $\Psi(\mu) = \tilde{X}$.

On the other hand, using that the map $\#_\Lambda : \Omega^1(G) \to \mathfrak{X}(G)$ is an isomorphism of $C^\infty(G,\mathbb{R})$-modules, we conclude that Ψ is an isomorphism of $C^\infty(M,\mathbb{R})$-modules. Note that $\Psi(f\mu) = (f\circ\alpha)\Psi(\mu)$, for $f \in C^\infty(M,\mathbb{R})$ and $\mu \in \Omega^1(M)$.

Now, denote by $([\![\,,\,]\!],\rho)$ the Lie algebroid structure on AG and suppose that $X, Y \in \Gamma(AG)$. We have that the left-invariant vector field $\overleftarrow{X}$ is α-projectable to the vector field $\rho(X)$. In addition, if μ and ν are 1-forms on M satisfying $\Psi(\mu) = \overleftarrow{X}$ and $\Psi(\nu) = \overleftarrow{Y}$ then, from Proposition 5.26 and since, $\sigma\circ\epsilon = 0$, it follows that

$$\rho(X) = \widetilde{\#}_{(\Lambda_0,E_0,\theta_0)}(\mu).$$

Using (5.53), we obtain that

$$\omega(\overleftarrow{X}) = \omega(\overleftarrow{Y}) = 0, \tag{5.95}$$

which implies that (see (1.11))

$$\begin{array}{rcl} i_{\overleftarrow{Y}}((\mathcal{L}_0)_{\overleftarrow{X}}\Omega) & = & e^\sigma(-\overleftarrow{Y}(\sigma)\alpha^*\mu + (\alpha^*\mu)(\overleftarrow{Y})(d_0\sigma - \omega) \\ & & +d_0((\alpha^*\mu)(\overleftarrow{Y})) - (\mathcal{L}_0)_{\overleftarrow{Y}}\alpha^*\mu). \end{array}$$

Moreover, since σ is multiplicative and $\overleftarrow{Y}$ is a left-invariant vector field, we deduce that

$$\overleftarrow{Y}(\sigma) = (\overleftarrow{Y}(\sigma)\circ\epsilon)\circ\alpha.$$

In addition, it is clear that

$$(\mathcal{L}_0)_{\overleftarrow{Y}}\alpha^*\mu = \alpha^*((\mathcal{L}_0)_{\rho(Y)}\mu)$$

and therefore,

$$\begin{array}{rcl} i_{\overleftarrow{Y}}((\mathcal{L}_0)_{\overleftarrow{X}}\Omega) & = & e^\sigma\alpha^*(-(\mathcal{L}_0)_{\rho(Y)}\mu - (Y(\sigma)\circ\epsilon)\mu \\ & & +\mu(\rho(Y))\theta_0 + d_0(\mu(\rho(Y)))). \end{array} \tag{5.96}$$

Furthermore, using that σ is multiplicative, we have that

$$\rho(Y) = \#_{\Lambda_0}(\nu). \tag{5.97}$$

On the other hand, from (5.53) and since $\#_\Lambda(\theta) = \mathcal{H}^{(\Lambda,E)}_{e^\sigma}$, it follows that

$$(\overleftarrow{Y}(\sigma) \circ \epsilon)(x) = -\nu(x)(\alpha_*^{\epsilon(x)}(\mathcal{H}^{(\Lambda,E)}_{e^\sigma}(\epsilon(x))))$$

which, by Proposition 5.26, implies that

$$\overleftarrow{Y}(\sigma) \circ \epsilon = -\nu(E_0). \tag{5.98}$$

Consequently, (see (5.96), (5.97) and (5.98)),

$$i_{\overleftarrow{Y}}((\mathcal{L}_0)_{\overleftarrow{X}}\Omega) = e^\sigma \alpha^*(-(\mathcal{L}_0)_{\#_{\Lambda_0}(\nu)}\mu + \nu(E_0)\mu - \Lambda_0(\mu,\nu)\theta_0 - d_0(\Lambda_0(\mu,\nu))).$$

Finally,

$$(\mathcal{L}_0)_{\overleftarrow{X}}(i_{\overleftarrow{Y}}\Omega) = -e^\sigma(\overleftarrow{X}(\sigma)\alpha^*\nu + (\mathcal{L}_0)_{\overleftarrow{X}}\alpha^*\nu)$$

and, using that $\overleftarrow{X}(\sigma) = -\mu(E_0) \circ \alpha$ and the fact that

$$(\mathcal{L}_0)_{\overleftarrow{X}}\alpha^*\nu = \alpha^*((\mathcal{L}_0)_{\rho(X)}\nu) = \alpha^*((\mathcal{L}_0)_{\#_{\Lambda_0}(\mu)}\nu - \mu(E_0)\nu)$$

we conclude that

$$(\mathcal{L}_0)_{\overleftarrow{X}}(i_{\overleftarrow{Y}}\Omega) = -e^\sigma\alpha^*((\mathcal{L}_0)_{\#_{\Lambda_0}(\mu)}\nu - \mu(E_0)\nu).$$

Thus,

$$\begin{aligned} \flat_\Omega([\overleftarrow{X}, \overleftarrow{Y}]) &= i_{[\overleftarrow{X},\overleftarrow{Y}]}\Omega = (\mathcal{L}_0)_{\overleftarrow{X}}(i_{\overleftarrow{Y}}\Omega) - i_{\overleftarrow{Y}}((\mathcal{L}_0)_{\overleftarrow{X}}\Omega) \\ &= -e^\sigma\alpha^*([\![\mu,\nu]\!]_{(\Lambda_0,E_0,\theta_0)}). \end{aligned}$$

This ends the proof of our result. QED

Remark 5.29 Let $(G \rightrightarrows M, \Omega)$ be a symplectic groupoid. Then, the Jacobi bialgebroid is a Lie bialgebroid (see Example 1 in Examples 5.27) and the Jacobi structure on M is Poisson (see Example 3.4.1 in Chapter 3). In addition, the 1-form θ_0 on M identically vanishes. Thus, AG is isomorphic to the cotangent Lie algebroid T^*M. This result was proved in [14].

Finally, let us describe the Jacobi bialgebroid associated with a l.c.s. groupoid $(G \rightrightarrows M, \Omega, \omega, \sigma)$.

Theorem 5.30 *Let $(G \rightrightarrows M, \Omega, \omega, \sigma)$ be a l.c.s. groupoid, AG be the Lie algebroid of G and ϕ_0 (respectively, X_0) the corresponding 1-cocycle on AG (respectively, A^*G). Then, the Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$ is isomorphic to the pair $((T^*M, -E_0), (TM^-, -\theta_0))$, where TM^- denotes the Lie algebroid structure on TM given by $[X,Y]_- = -[X,Y]$ and $-Id(X) = -X$, for $X, Y \in \mathfrak{X}(M)$.*

Proof: If $\Psi : T^*M \to AG$ is the isomorphism defined in Theorem 5.28 then

$$\Psi(\nu_x) = \#_\Lambda((\alpha_*^{\epsilon(x)})^*(\nu_x), \text{ for } \nu_x \in T_x^*M. \tag{5.99}$$

Thus,

$$\Psi^*(\mu_x) = -\alpha_*^{\epsilon(x)}(\#_\Lambda(\tilde{\epsilon}(\mu_x))), \text{ for } \mu_x \in A_x^*G.$$

Therefore, using (5.80), the fact that $\widetilde{\#}_{(\Lambda,E)}[\![(\tilde{\mu},0),(\tilde{\nu},0)]\!]_{(\Lambda,E)} = [\#_\Lambda(\tilde{\mu}), \#_\Lambda(\tilde{\nu})]$, for $\tilde{\mu}, \tilde{\nu} \in \Omega^1(G)$ and since $\epsilon(M)$ is a coisotropic submanifold of G, we conclude that Ψ^* is also a Lie algebroid morphism.

On the other hand, from (5.93), it follows that

$$\epsilon^*(\beta^*(\theta_0)) = -\epsilon^*(\omega),$$

which implies that

$$\theta_0 = -\epsilon^*(\omega). \tag{5.100}$$

Finally, using (5.64), (5.99), (5.100) and since $E = -\overrightarrow{X}_0$, we obtain that

$$\Psi(\theta_0) = -X_0.$$

QED

4.-*Jacobi-Lie groups*

Let G be a Lie group with identity element $\mathfrak{e}$, $\sigma : G \to \mathbb{R}$ be a multiplicative function and (Λ, E) be a Jacobi structure on G such that Λ is σ-multiplicative, E is a right-invariant vector field and

$$\#_\Lambda(d_0\sigma)(g) = -E(g) + e^{-\sigma(g)}(L_g)_*^{\mathfrak{e}}(E(\mathfrak{e})),$$

for all $g \in G$. Then, $(G \rightrightarrows \{\mathfrak{e}\}, \Lambda, E, \sigma)$ is a Jacobi groupoid (see Section 5.3.4) and the corresponding Jacobi bialgebroid is a Jacobi bialgebra. In fact, the Lie algebroid of G is just the Lie algebra $\mathfrak{g}$ of G, that is, $AG = \mathfrak{g}$ and, from (5.79), it follows that $\phi_0 = (d_0\sigma)(\mathfrak{e})$.

On the other hand, since $\Lambda(\mathfrak{e}) = 0$, one may consider the intrinsic derivative $\delta_{\mathfrak{e}}\Lambda : \mathfrak{g} \to \wedge^2\mathfrak{g}$ of Λ at $\mathfrak{e}$. Moreover, using (1.29) and (5.80), we deduce that the Lie bracket $[\, , \,]_*$ on the dual space $A^*G = \mathfrak{g}^*$ of $\mathfrak{g}$ is given by

$$[\mu, \nu]_* = [\mu, \nu]_\Lambda - \mu(E(\mathfrak{e}))\nu + \nu(E(\mathfrak{e}))\mu$$

for $\mu, \nu \in \mathfrak{g}^*$, where $[\, , \,]_\Lambda : \mathfrak{g}^* \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the adjoint map of the intrinsic derivative of Λ at $\mathfrak{e}$. In addition, the 1-cocycle X_0 on $\mathfrak{g}^*$ is $X_0 = -E(\mathfrak{e})$.

5.-*An abelian Jacobi groupoid*

Let $(L, [\![\, , \,]\!], \rho)$ be a Lie algebroid over a manifold M and $\mu_0 \in \Gamma(L^*)$ be a 1-cocycle of L. We may consider on L^* the Jacobi structure $(\Lambda_{(L^*,\mu_0)}, E_{(L^*,\mu_0)})$ given by (2.7) and the Lie groupoid structure for which $\alpha = \beta$ is the vector bundle projection $\tau : L^* \to M$ and the partial multiplication is the addition in the fibers. As we know (see Section 5.3.5), $(L^* \rightrightarrows M, \Lambda_{(L^*,\mu_0)}, E_{(L^*,\mu_0)}, 0)$ is a Jacobi groupoid and therefore we have the corresponding Jacobi bialgebroid $((A(L^*), \phi_0 = 0), (A^*(L^*), X_0))$.

On the other hand, if $0 : M \to L^*$ is the zero section of L^* and $\mu \in \tau^{-1}(x) = L^*_x$, we will denote by $\mu^{\mathbf{v}}(0(x)) \in T_{0(x)}L^*_x$ the vertical lift of μ to L^* at the point $0(x)$. Then, the map

$$\mathbf{v} : L^* \to A(L^*), \quad \mu \in L^*_x \mapsto \mu^{\mathbf{v}}(0(x)) \in A_x(L^*),$$

defines an isomorphism between the vector bundles L^* and $A(L^*)$. Moreover, using (2.7) and since $\alpha = \tau$ and the Lie bracket of two left-invariant vector fields on L^* is zero, we conclude that:

i) $\mathbf{v}$ defines an isomorphism between the Lie algebroid L^* (with the trivial Lie algebroid structure), and $A(L^*)$ and

ii) $\mathbf{v}(\mu_0) = X_0$.

In addition, if $\mathbf{v}^* : A^*(L^*) \to L$ is the adjoint map of $\mathbf{v} : L^* \to A(L^*)$ then, from (1.29), (1.42), (2.7) and (5.80), we deduce that $\mathbf{v}^*$ induces an isomorphism between the Lie algebroids $A^*(L^*)$ and $(L, [\![\,,\,]\!], \rho)$.

Therefore, we have proved that the Jacobi bialgebroids $((A(L^*), 0), (A^*(L^*), X_0))$ and $((L^*, 0), (L, \mu_0))$ are isomorphic.

6.- *The banal Jacobi groupoid*

Let (M, Λ, E) be a Jacobi manifold and G the product manifold $M \times \mathbb{R} \times M$. Denote by (Λ', E') the Jacobi structure on G given by (5.71) and by $\sigma : G \to \mathbb{R}$ the function defined by $\sigma(x, t, y) = t$. Then, one may consider a Lie groupoid structure in G over M in such a way that $(G \rightrightarrows M, \Lambda', E', \sigma)$ is a Jacobi groupoid (see Section 5.3.6). Thus, we have the corresponding Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$. As we know, the map $\Phi : TM \times \mathbb{R} \to AG$ given by (5.70) defines an isomorphism between the Lie algebroids $(TM \times \mathbb{R}, [\;,\;], \pi)$ and AG and, moreover, it follows that $\Phi(-E, 0) = X_0$.

Now, let $\Phi^* : A^*G \to T^*M \times \mathbb{R}$ be the adjoint map of Φ. Then, using (1.29), (5.70), (5.71) and (5.79), we deduce that Φ^* induces an isomorphism between the Lie algebroids A^*G and $(T^*M \times \mathbb{R}, [\![\,,\,]\!]_{(\Lambda,E)}, \widetilde{\#}_{(\Lambda,E)})$ and, in addition, $\Phi^*(\phi_0) = (0, 1)$.

Therefore, we have proved that the Jacobi bialgebroids $((AG, \phi_0), (A^*G, X_0))$ and $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)))$ are isomorphic.

5.4.3 Integration of Jacobi bialgebroids

In this Section, we will show a converse of Theorem 5.25, that is, we will show that one may integrate a Jacobi bialgebroid and obtain a Jacobi groupoid.

Jacobi groupoids and Poisson groupoids

In this first part, we will prove that a Poisson groupoid can be obtained from any Jacobi groupoid and we will show the relation between the Jacobi bialgebroid associated with the Jacobi groupoid and the Lie bialgebroid induced by the Poisson groupoid.

Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Then, using the multiplicative character of σ, we have defined a right action of $G \rightrightarrows M$ on the canonical projection $\pi_1 : M \times \mathbb{R} \to M$ by (5.56) and the corresponding action groupoid may be identified with $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$ with the structural functions given by (5.59). Moreover, the Lie algebroid of the Lie groupoid $G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}$ may be identified with $(AG \times \mathbb{R}, [\![\, , \,]\!]^{\bar{}\phi_0}, \bar{\rho}^{\phi_0})$, where ϕ_0 is the 1-cocycle on AG given by (5.57).

We also have the following result.

Proposition 5.31 *Let $G \rightrightarrows M$ be a Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Suppose that (Λ, E) is a Jacobi structure on G, that $\Pi = e^{-t}(\Lambda + \frac{\partial}{\partial t} \wedge E)$ is the Poissonization on $G \times \mathbb{R}$ and that in $G \times \mathbb{R}$ we consider the Lie groupoid structure on $M \times \mathbb{R}$ with structural functions given by (5.59). Then, $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid if and only if $(G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}, \Pi)$ is a Poisson groupoid.*

Proof: From (1.19), we have that the homomorphism $\#_\Pi : T^*(G \times \mathbb{R}) \to T(G \times \mathbb{R})$ is given by

$$\#_\Pi(\mu_g + \gamma d_0 t_{|t}) = e^{-t}\Big(\#_\Lambda(\mu_g) + \gamma E(g) - \mu_g(E(g))\frac{\partial}{\partial t}_{|t}\Big), \tag{5.101}$$

for $\mu_g + \gamma d_0 t_{|t} \in T^*_{(g,t)}(G \times \mathbb{R})$.

Now, we consider in $T^*G \times \mathbb{R}$ (respectively, $TG \times \mathbb{R}$) the Lie groupoid structure over A^*G (respectively, $TM \times \mathbb{R}$) with structural functions defined by (5.15) (respectively, (5.8)). Then, an straightforward computation, using (1.55), (1.58), (5.8), (5.10), (5.15), (5.61), (5.62) and (5.101), shows that $\#_{(\Lambda,E)} : T^*G \times \mathbb{R} \to TG \times \mathbb{R}$ is a Lie groupoid morphism over some map $\varphi_0 : A^*G \to TM \times \mathbb{R}$ if and only if $\#_\Pi : T^*(G \times \mathbb{R}) \to T(G \times \mathbb{R})$ is a Lie groupoid morphism over some map $\psi_0 : A^*G \times \mathbb{R} \to T(M \times \mathbb{R})$. This proves the result.

QED

Let $((A, \phi_0), (A^*, X_0))$ be a Jacobi bialgebroid and denote by $([\![\, , \,]\!], \rho)$ (respectively, $([\![\, , \,]\!]_*, \rho_*)$) the Lie algebroid structure on A (respectively, A^*). Then, if on the vector bundle $A\times\mathbb{R}\to M\times\mathbb{R}$ (respectively, $A^*\times\mathbb{R}\to M\times\mathbb{R}$) we consider the Lie algebroid structure $([\![\;,\;]\!]^{\bar{\phi}_0},\bar{\rho}^{\phi_0})$ (respectively, $([\![\;,\;]\!]_*^{\hat{X}_0},\hat{\rho}_*^{X_0})$) then the pair $(A\times\mathbb{R},A^*\times\mathbb{R})$ is a Lie bialgebroid (see Theorem 3.29). In particular, if $(G\rightrightarrows M,\Lambda,E,\sigma)$ is a Jacobi groupoid and AG is the Lie algebroid of G then the pair $(AG\times\mathbb{R},A^*G\times\mathbb{R})$ is a Lie bialgebroid. Furthermore, we have

Proposition 5.32 *Let $(G\rightrightarrows M,\Lambda,E,\sigma)$ be a Jacobi groupoid and $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\Pi)$ be the corresponding Poisson groupoid. If $((AG,\phi_0),(A^*G,\,X_0))$ (respectively, $(A(G\times\mathbb{R}),A^*(G\times\mathbb{R}))$) is the Jacobi bialgebroid (respectively, the Lie bialgebroid) associated with $(G\rightrightarrows M,\Lambda,E,\sigma)$ (respectively, $(G\times\mathbb{R}\rightrightarrows M\times\mathbb{R},\Pi)$), then the Lie bialgebroids $(A(G\times\mathbb{R}),A^*(G\times\mathbb{R}))$ and $(AG\times\mathbb{R},A^*G\times\mathbb{R})$ are isomorphic.*

Proof: Denote by $([\![\;,\;]\!],\rho)$ and $([\![\;,\;]\!]_*,\rho_*)$ the Lie algebroid structures on AG and A^*G, respectively, and by $\mathcal{J}:A(G\times\mathbb{R})\to AG\times\mathbb{R}$ the isomorphism between the Lie algebroids $A(G\times\mathbb{R})$ and $(AG\times\mathbb{R},[\![\;,\;]\!]^{\bar{\phi}_0},\bar{\rho}^{\phi_0})$ given by (5.60).

Now, let $\tilde{\mathcal{J}}^*:T^*G\times\mathbb{R}\times\mathbb{R}\to T^*(G\times\mathbb{R})$ be the map defined by

$$\tilde{\mathcal{J}}^*(\mu_g,\gamma,t)=\mu_g+\gamma\,d_0t_{|t},$$

for $\mu_g\in T_g^*G$ and $\gamma,t\in\mathbb{R}$.

If we identify A^*G (respectively, $A^*(G\times\mathbb{R})$) with the conormal bundle of $\epsilon(M)$ (respectively, $\epsilon_\sigma(M\times\mathbb{R})$) then the restriction of $\tilde{\mathcal{J}}^*$ to $A^*G\times\{0\}\times\mathbb{R}\cong A^*G\times\mathbb{R}$ is just the adjoint operator $\mathcal{J}^*:A^*G\times\mathbb{R}\to A^*(G\times\mathbb{R})$ of $\mathcal{J}$. Therefore, from (2.12), (3.87), (5.80) and Remark 5.24, we conclude that the map $\mathcal{J}^*$ is an isomorphism between the Lie algebroids $(A^*G\times\mathbb{R},[\![\;,\;]\!]_*^{\hat{X}_0},\hat{\rho}_*^{X_0})$ and $A^*(G\times\mathbb{R})$. QED

Integration of Jacobi bialgebroids

Next, we will show a converse of Theorem 5.25.

For this purpose, we will use the notion of the derivative of an affine k-vector field on a Lie groupoid (see [85]). Let G be a Lie groupoid with Lie algebroid AG and P be an affine k-vector field on G. Then, the derivative of P, δP, is the map $\delta P : \Gamma(AG) \to \Gamma(\wedge^k(AG))$ defined as follows. If $X \in \Gamma(AG)$, $\delta P(X)$ is the element in $\Gamma(\wedge^k(AG))$ whose left translation is $(\mathcal{L}_0)_{\overleftarrow{X}} P$.

Now, we will prove the announced result at the beginning of this Section.

Theorem 5.33 *Let $((AG, \phi_0), (A^*G, X_0))$ be a Jacobi bialgebroid where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$. Then, there is a unique multiplicative function $\sigma : G \to \mathbb{R}$ and a unique Jacobi structure (Λ, E) on G that makes $(G \rightrightarrows M, \Lambda, E, \sigma)$ into a Jacobi groupoid with Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$.*

Proof: Since G is α-connected and α-simply connected, we deduce that there exists a unique multiplicative function $\sigma : G \to \mathbb{R}$ such that

$$\phi_0(X) = X(\sigma),$$

for all $X \in \Gamma(AG)$. The multiplicative function $\sigma : G \to \mathbb{R}$ allows us to construct a Lie groupoid structure in $G \times \mathbb{R}$ over $M \times \mathbb{R}$ with structural functions $\alpha_\sigma, \beta_\sigma, m_\sigma$ and ϵ_σ given by (5.59).

If $([\![\ ,\ ]\!], \rho)$ is the Lie algebroid structure on AG then, as we know, the Lie algebroid of $G \times \mathbb{R}$ is $(AG \times \mathbb{R}, [\![\ ,\ ]\!]^{\bar{\phi}_0}, \bar{\rho}^{\phi_0})$. Moreover, if $([\![\ ,\ ]\!]_*, \rho_*)$ is the Lie algebroid structure on A^*G and we consider on the vector bundle $A^*G \times \mathbb{R} \to M \times \mathbb{R}$ the Lie algebroid structure $([\![\ ,\ ]\!]_*^{\hat{}X_0}, \hat{\rho}_*^{X_0})$ given by (2.12), it follows that the pair $(AG \times \mathbb{R}, A^*G \times \mathbb{R})$ is a Lie bialgebroid. Therefore, using Theorem 4.1 in [85], we obtain that there is a unique Poisson structure Π on $G \times \mathbb{R}$ that makes $G \times \mathbb{R}$ into a Poisson groupoid with Lie bialgebroid $(AG \times \mathbb{R}, A^*G \times \mathbb{R})$. Thus, Π is affine.

We will see that the 2-vector (on $G \times \mathbb{R}$) $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}} \Pi + \Pi$ is affine, where $\bar{\mathcal{L}}_0$ is the Lie derivative on $G \times \mathbb{R}$. For this purpose, we will use the following relation

$$(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}} \overleftarrow{\bar{P}} = \overleftarrow{\frac{\partial \bar{P}}{\partial t}}, \tag{5.102}$$

for $\bar{P} \in \Gamma(\wedge^k(AG \times \mathbb{R}))$. Note that $\bar{P}$ is a time-dependent section of the vector bundle $\wedge^k(AG) \to M$ and, thus, one may consider the derivative of $\bar{P}$ with respect to the time, $\dfrac{\partial \bar{P}}{\partial t}$.

From (5.102) and Proposition 5.17, we conclude that the vector field $\dfrac{\partial}{\partial t}$ is affine. Consequently (see Proposition 2.5 in [85]), the 2-vector $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi$ is also affine.

Next, we will show that the Poisson structure Π is homogeneous with respect to the vector field $\dfrac{\partial}{\partial t}$. This fact implies that Π is the Poissonization of a Jacobi structure (Λ, E) on G (see Section 1.1.6). Moreover, from Propositions 5.31 and 5.32, we will have that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid with Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$.

Therefore, we must prove that Π is homogeneous. Now, using Theorem 2.6 in [85] and since G is α-connected and the 2-vector $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi$ is affine, we deduce that Π is homogeneous if and only if:

i) The derivative of the 2-vector $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi$ is zero and

ii) The restriction of the 2-vector $(\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi$ to the points of $\epsilon_\sigma(M \times \mathbb{R})$ is zero.

First, we will show *i)*. If H' is a Poisson groupoid with Poisson structure Π' and Lie algebroid AH', we have that (see Theorem 3.1 in [120])

$$(\mathcal{L}_0)_{\overleftarrow{X}}\Pi' = -\overleftarrow{d_* X}, \tag{5.103}$$

for $X \in \Gamma(AH')$, where d_* is the differential of the dual Lie algebroid A^*H'. Thus, from (5.102) and (5.103), it follows that

$$\begin{aligned} (\bar{\mathcal{L}}_0)_{\overleftarrow{\bar{X}}}\Big((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi\Big) &= (\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}(\bar{\mathcal{L}}_0)_{\overleftarrow{\bar{X}}}\Pi - (\bar{\mathcal{L}}_0)_{\overleftarrow{\frac{\partial \bar{X}}{\partial t}}}\Pi + (\bar{\mathcal{L}}_0)_{\overleftarrow{\bar{X}}}\Pi \\ &= \overleftarrow{\hat{d}_*^{X_0}\frac{\partial \bar{X}}{\partial t}} - \overleftarrow{\hat{d}_*^{X_0}\bar{X}} - \frac{\overleftarrow{\partial(\hat{d}_*^{X_0}\bar{X})}}{\partial t}, \end{aligned}$$

for $\bar{X} \in \Gamma(AG \times \mathbb{R})$. On the other hand, using (3.83), we obtain that

$$\widehat{d_*}^{X_0}\bar{Z} = e^{-t}\Big(d_*\bar{Z} + X_0 \wedge (\bar{Z} + \frac{\partial \bar{Z}}{\partial t})\Big),$$

for $\bar{Z} \in \Gamma(AG \times \mathbb{R})$, d_* also denoting the differential of the Lie algebroid $(A^*G \times \mathbb{R}, [\![\ ,\ ]\!]_*, \rho_*)$. Consequently, we deduce that

$$(\bar{\mathcal{L}}_0)_{\overleftarrow{\bar{X}}}\Big((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi\Big) = 0.$$

Next, we will show *ii)*. If (x,t) is a point of $M \times \mathbb{R}$ then

$$T^*_{\epsilon_\sigma(x,t)}(G \times \mathbb{R}) \cong A^*_{(x,t)}(G \times \mathbb{R}) \oplus ((\alpha_\sigma)_*^{(\epsilon(x),t)})^*(T^*_{(x,t)}(M \times \mathbb{R})).$$

Therefore, if we denote by d_0 the usual differential on $G \times \mathbb{R}$ and $M \times \mathbb{R}$, it is enough to prove that

$$((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi)(d_0F_1, d_0F_2)_{|\epsilon_\sigma(M\times\mathbb{R})} = 0,$$

when F_1 and F_2 are either constant on $\epsilon_\sigma(M \times \mathbb{R})$ or equal to $(\alpha_\sigma)^*f_i$, with $f_i \in C^\infty(M \times \mathbb{R}, \mathbb{R})$, $i = 1, 2$. We will distinguish three cases:

First case. Suppose that $F_1 = (\alpha_\sigma)^*f_1$ and $F_2 = (\alpha_\sigma)^*f_2$, with $f_1, f_2 \in C^\infty(M \times \mathbb{R}, \mathbb{R})$. Denote by Π_0 the Poisson structure on $M \times \mathbb{R}$ induced by the Lie bialgebroid $(AG \times \mathbb{R}, A^*G \times \mathbb{R})$ and by $\{\ ,\ \}_\Pi$ (respectively, $\{\ ,\ \}_{\Pi_0}$) the Poisson bracket on $G \times \mathbb{R}$ (respectively, $M \times \mathbb{R}$) associated with Π (respectively, Π_0). Then, from Proposition 5.31 and since the vector field $\dfrac{\partial}{\partial t}$ on $G \times \mathbb{R}$ is α_σ-projectable, it follows that

$$\begin{aligned}((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi)(d_0F_1, d_0F_2) \;=\; & \alpha_\sigma^*\Big(\frac{\partial}{\partial t}\{f_1, f_2\}_{\Pi_0} - \{\frac{\partial f_1}{\partial t}, f_2\}_{\Pi_0} \\ & -\{f_1, \frac{\partial f_2}{\partial t}\}_{\Pi_0} + \{f_1, f_2\}_{\Pi_0}\Big).\end{aligned}$$

Thus, using that the Poisson structure Π_0 is homogeneous with respect to the vector field $\dfrac{\partial}{\partial t}$ on $M \times \mathbb{R}$ (see Theorem 3.29), we obtain that

$$((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}}\Pi + \Pi)(d_0F_1, d_0F_2) = 0.$$

Second case. Suppose that $F_1 = (\alpha_\sigma)^* f_1$, with $f_1 \in C^\infty(M \times \mathbb{R}, \mathbb{R})$ and that F_2 is constant on $\epsilon_\sigma(M \times \mathbb{R})$. Following the proof of Lemma 4.12 in [85], we deduce that

$$\{(\alpha_\sigma)^* f, H\}_\Pi = \overleftarrow{((\hat{\rho}_*^{X_0})^*(d_0 f))}(H), \tag{5.104}$$

for $f \in C^\infty(M \times \mathbb{R}, \mathbb{R})$ and $H \in C^\infty(G \times \mathbb{R}, \mathbb{R})$. Note that (see (2.12))

$$(\hat{\rho}_*^{X_0})^*(\mu + g d_0 t) = e^{-t}((\rho_*)^*(\mu) + g X_0), \tag{5.105}$$

for $g \in C^\infty(M \times \mathbb{R}, \mathbb{R})$ and μ a time-dependent 1-form on M. Therefore, from (5.102), (5.104), (5.105) and since $\dfrac{\partial F_2}{\partial t} = 0$, we have that

$$\begin{aligned}
((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}} \Pi + \Pi)(d_0 F_1, d_0 F_2) &= -[\frac{\partial}{\partial t}, \overleftarrow{(\hat{\rho}_*^{X_0})^*(d_0 f_1)}](F_2) - \overleftarrow{(\hat{\rho}_*^{X_0})^*(d_0(\frac{\partial f_1}{\partial t}))}(F_2) \\
&\quad - \overleftarrow{(\hat{\rho}_*^{X_0})^*(d_0 f_1)}(F_2) \\
&= \overleftarrow{\frac{\partial}{\partial t}((\hat{\rho}_*^{X_0})^*(d_0 f_1))} - \overleftarrow{(\hat{\rho}_*^{X_0})^*(d_0(\frac{\partial f_1}{\partial t}))} \\
&\quad - \overleftarrow{(\hat{\rho}_*^{X_0})^*(d_0 f_1)}(F_2) = 0.
\end{aligned}$$

Third case. Suppose that F_1 and F_2 are constant on $\epsilon_\sigma(M \times \mathbb{R})$. Then, using that $\epsilon_\sigma(M \times \mathbb{R})$ is a coisotropic submanifold of $(G \times \mathbb{R}, \Pi)$, it follows that

$$\{F_1, F_2\}_{\Pi|\epsilon_\sigma(M \times \mathbb{R})} = 0.$$

Moreover, since $\dfrac{\partial F_1}{\partial t} = \dfrac{\partial F_2}{\partial t} = 0$ and the restriction to $\epsilon_\sigma(M \times \mathbb{R})$ of the vector field $\dfrac{\partial}{\partial t}$ is tangent to $\epsilon_\sigma(M \times \mathbb{R})$, we conclude that

$$((\bar{\mathcal{L}}_0)_{\frac{\partial}{\partial t}} \Pi + \Pi)(d_0 F_1, d_0 F_2)_{|\epsilon_\sigma(M \times \mathbb{R})} = 0.$$

QED

Examples 5.34 1.- *Lie bialgebroids*

Let (AG, A^*G) be a Lie bialgebroid where AG is the Lie algebroid of an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$. Then, using Theorem 5.33, we obtain that there exists a unique Poisson structure Λ on

G that makes $(G \rightrightarrows M, \Lambda)$ into a Poisson groupoid with Lie bialgebroid (AG, A^*G). This result was proved in [85].

2.- *Jacobi bialgebras*
Let $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$ be a Jacobi bialgebra and G be a connected simply connected Lie group with Lie algebra $\mathfrak{g}$. Then, using Theorem 5.33 (see also Section 5.3.4), we deduce that there exists a unique multiplicative function $\sigma : G \to \mathbb{R}$ and a unique Jacobi structure (Λ, E) on G such that (G, Λ, E, σ) is a Jacobi-Lie group with Jacobi bialgebra $((\mathfrak{g}, \phi_0), (\mathfrak{g}^*, X_0))$.

3.- *Integration of Jacobi manifolds*
Suppose that (M, Λ_0, E_0) is a Jacobi manifold and that $(T^*M \times \mathbb{R}, [\![\, , \,]\!]_{(\Lambda_0, E_0)}, \widetilde{\#}_{(\Lambda_0, E_0)})$ is the corresponding Lie algebroid. Moreover, assume that there exists an α-connected and α-simply connected Lie groupoid $G \rightrightarrows M$ with Lie algebroid $AG = T^*M \times \mathbb{R}$.

Then, the pair $((A = T^*M \times \mathbb{R}, X_0), (A^* = TM \times \mathbb{R}, \phi_0))$ is a Jacobi bialgebroid, where $X_0 = (-E_0, 0) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$ and $\phi_0 = (0, 1) \in \Omega^1(M) \times C^\infty(M, \mathbb{R})$ (see Section 3.4.2 and Theorem 3.31). Thus, using Theorem 5.33, we obtain that there exists a unique multiplicative function σ on G and a unique Jacobi structure (Λ, E) on G such that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid with Jacobi bialgebroid $((A, \phi_0), (A^*, X_0))$.

On the other hand, if on the vector bundle $\bar{A} = A \times \mathbb{R} \to M \times \mathbb{R}$ (respectively, $\bar{A}^* = A^* \times \mathbb{R} \to M \times \mathbb{R}$) we consider the Lie algebroid structure $([\![\, , \,]\!]^{\bar{}X_0}_{(\Lambda_0, E_0)}, (\widetilde{\#}_{(\Lambda_0, E_0)})^{\bar{}X_0})$ (respectively, $([\ ,\ ]^{\hat{}\phi_0}, \hat{\pi}^{\phi_0})$ then, from Theorem 3.29, we deduce that $(\bar{A}, \bar{A}^*)$ is a Lie bialgebroid.

Now, denote by Π_0 the Poissonization on $M \times \mathbb{R}$ of the Jacobi structure (Λ_0, E_0). Then, the map $\Upsilon : \bar{A} \to T^*(M \times \mathbb{R})$ defined by

$$\Upsilon(\mu_x, \lambda, t) = e^t(\mu_x + \lambda \, d_0 t_{|t}),$$

for $\mu_x \in T^*_x M$ and $\lambda, t \in \mathbb{R}$, induces an isomorphism between the Lie bialgebroid $(\bar{A}, \bar{A}^*)$ and the Lie bialgebroid $(T^*(M \times \mathbb{R}), T(M \times \mathbb{R}))$ associated with the Poissonization Π_0 of (Λ_0, E_0) (see Section 3.5.2).

In addition, as we know (see Proposition 5.31), the product manifold $G \times \mathbb{R}$ is a Lie groupoid on $M \times \mathbb{R}$ and if Π is the Poissonization on $G \times \mathbb{R}$ of (Λ, E), we have that $(G \times \mathbb{R} \rightrightarrows M \times \mathbb{R}, \Pi)$ is a Poisson groupoid with associated Lie bialgebroid $(\bar{A}, \bar{A}^*) \cong (T^*(M \times \mathbb{R}), T(M \times \mathbb{R}))$. Therefore, using Theorem 5.3 in [85], we get that Π is a symplectic structure which implies that (Λ, E) is a Jacobi structure induced by a contact 1-form η on G (see Examples 1.5).

So, we conclude that given a Jacobi manifold (M, Λ_0, E_0) there always exists, at least locally, a contact groupoid $(G \rightrightarrows M, \eta, \sigma)$ such that $AG \cong T^*M \times \mathbb{R}$. This result was first proved in [23] (see also [2]). We remark that in [20] has been shown in which conditions a Jacobi manifold can be integrated to a contact groupoid.

4.- *Triangular Jacobi bialgebroids*

Let $G \rightrightarrows M$ be an α-connected Lie groupoid and $\sigma : G \to \mathbb{R}$ be a multiplicative function. Moreover, let us consider the associated Lie algebroid $(AG, [\![\ ,\ ]\!], \rho)$ and the 1-cocycle ϕ_0 associated with σ (see (5.79)). Suppose that $\mathcal{C}$ is a ϕ_0-canonical section, that is, $\mathcal{C} \in \Gamma(\wedge^2 A)$ and

$$[\![\mathcal{C}, \mathcal{C}]\!]^{\phi_0} = 0.$$

Then, we know that there exists a Lie algebroid structure $([\![\ ,\ ]\!]_{*\mathcal{C}}, \rho_{*\mathcal{C}})$ on A^*G and a 1-cocycle $X_0 = -\#_{\mathcal{C}}(\phi_0)$ such that $((AG, \phi_0), (A^*G, X_0))$ is a Jacobi bialgebroid (see Theorem 3.20).

Now, we introduce the 2-vector Λ and the vector field E defined by

$$\Lambda = e^{-\sigma}\overleftarrow{C} - \overrightarrow{C}, \qquad E = -\overrightarrow{X}_0 = \overrightarrow{\#_{\mathcal{C}}(\phi_0)}, \tag{5.106}$$

Using (1.53), Proposition 5.17 and the fact that σ is a multiplicative function, we have that Λ is a σ-affine 2-vector field on G. Moreover, following the proof of Theorem 3.1 in [79], it is not difficult to show that the following properties hold:

$$
\begin{aligned}
&\beta_*(\Lambda(g)) = -\rho(C)(\beta(g)),\\
&\alpha_*(\Lambda(g)) = e^{-\sigma(g)}\rho(C)(\alpha(g)),\\
&\Lambda(\alpha^* f_1, \beta^* f_2) = 0, \qquad (5.107)\\
&i_{e^\sigma d_0(\alpha^* f)}\Lambda \text{ is a left-invariant vector field on G,}\\
&i_{d_0(\beta^* f)}\Lambda \text{ is a right-invariant vector field on G,}
\end{aligned}
$$

for $g \in G$ and $f, f_1, f_2 \in C^\infty(M, \mathbb{R})$.

On the other hand, denote by $AD(G)$ the affinoid diagram of G (see the proof of Proposition 5.15) and suppose that g and h are composable elements of G such that $\alpha(g) = \beta(h) = x$. Then, $(gh, g, h, \epsilon(x))$ is an element of $AD(G)$ and the following three types of covectors are conormal to $AD(G)$ at the point $(gh, g, h, \epsilon(x))$:

$$
\begin{aligned}
&(-\xi, ((R_{\mathcal{Y}})^g_*)^*(\xi), ((L_{\mathcal{X}})^h_*)^*(\xi), -((R_{\mathcal{Y}} \circ L_{\mathcal{X}})^{\epsilon(x)}_*)^*(\xi)),\\
&(-(\beta^{gh}_*)^*(\eta), (\beta^g_*)^*(\eta), 0, 0),\\
&(-(\alpha^{gh}_*)^*(\mu), 0, (\alpha^h_*)^*(\mu), 0),
\end{aligned}
$$

with $\xi \in T^*_{gh}G$, $\eta \in T^*_{\beta(g)}M$ and $\mu \in T^*_{\alpha(h)}M$, where $\mathcal{X}$ and $\mathcal{Y}$ are (local) bisections through the points g and h ($\mathcal{X}(x) = g$ and $\mathcal{Y}(x) = h$). In fact, these covectors span the whole conormal space of $AD(G)$ (see the proof of Theorem 2.8 in [120]). Using this fact, (5.107) and since Λ is a σ-affine 2-vector field on G, we deduce that $AD(G)$ is a coisotropic submanifold of $G \times G \times G \times G$ with respect to the 2-vector $\tilde{\tilde{\Xi}}$ on $G \times G \times G \times G$, which is given by

$$
\tilde{\tilde{\Xi}}(k, g, h, r) = e^{\sigma(k)}\Lambda(k) - e^{\sigma(k)}\Lambda(g) - e^{\sigma(h)}\Lambda(h) + e^{\sigma(h)}\Lambda(r).
$$

Moreover, if $\nu_1, \nu_2 \in T_{\epsilon(x)}(\epsilon(M))^\circ$, the conormal space to $T_{\epsilon(x)}(\epsilon(M))$, we get that

$$
\Lambda(\epsilon(x))(\nu_1, \nu_2) = 0
$$

and, therefore M is a coisotropic submanifold of G with respect to Λ. Thus, the classical techniques of coisotropic calculus (see Theorem 4.5 in [118]) allow us to deduce that the graph of the multiplication in G, $\{(g,h,gh) \in G \times G \times G \,/\, \alpha(g) = \beta(h)\}$, is a coisotropic submanifold of $G \times G \times G$ with respect to Ξ on $G \times G \times G$ given by

$$\Xi(g,h,k) = e^{\sigma(g)}\Lambda(g) + \Lambda(h) - e^{\sigma(g)}\Lambda(k).$$

This implies that the map $\#_\Lambda : T^*G \to TG$ is a Lie groupoid morphism from the σ-cotangent groupoid $T^*G \rightrightarrows A^*G$ to the tangent Lie groupoid $TG \rightrightarrows TM$.

Now, using that σ is multiplicative, we have that

$$\#_{\overleftarrow{\mathcal{C}}}(d_0\sigma) = \overleftarrow{i_{\phi_0}\mathcal{C}}, \qquad \#_{\overrightarrow{\mathcal{C}}}(d_0\sigma) = \overrightarrow{i_{\phi_0}\mathcal{C}}. \tag{5.108}$$

As a consequence,

$$\begin{aligned} \#_\Lambda(d_0\sigma) &= \#_{e^{-\sigma}\overleftarrow{\mathcal{C}}-\overrightarrow{\mathcal{C}}}(d_0\sigma) \\ &= e^{-\sigma}\overleftarrow{i_{\phi_0}\mathcal{C}} - \overrightarrow{i_{\phi_0}\mathcal{C}} \\ &= \overrightarrow{X}_0 - e^{-\sigma}\overleftarrow{X}_0. \end{aligned} \tag{5.109}$$

Finally, let us show that (Λ, E) is a Jacobi structure on G. From (1.53), (3.22), (5.106), (5.108) and (5.109), we have that

$$[\Lambda,\Lambda] - 2E \wedge \Lambda = -\overrightarrow{[\![\mathcal{C},\mathcal{C}]\!]^{\phi_0}} + e^{-2\sigma}\overleftarrow{[\![\mathcal{C},\mathcal{C}]\!]^{\phi_0}}$$

Thus, since $\mathcal{C}$ is a ϕ_0-canonical section, we deduce that $[\Lambda,\Lambda] = 2E\wedge\Lambda$.

On the other hand, using (5.106) and the fact that $\phi_0(X_0) = 0$, we get that $E(\sigma) = 0$. Therefore, from (1.53), (3.22), (3.24) and (5.106), we deduce that

$$[E,\Lambda] = -\frac{1}{2}\overrightarrow{i_{\phi_0}[\![\mathcal{C},\mathcal{C}]\!]} = -\frac{1}{2}\overrightarrow{i_{\phi_0}[\![\mathcal{C},\mathcal{C}]\!]^{\phi_0}} = 0.$$

Consequently, using Theorem 5.13, it follows that (Λ, E) is a Jacobi structure such that $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid. This implies that the dual vector bundle to AG, A^*G, admits a Lie algebroid structure $([\![\,,\,]\!]_*, \rho_*)$ and, in addition, X_0 is a 1-cocycle of $(A^*G, [\![\,,\,]\!]_*, \rho_*)$ and the X_0-differential of $d_*^{X_0}$ of A^*G is given by $\overleftarrow{d_*^{X_0}X} = -e^{\sigma}[\overleftarrow{X},\Lambda]$, for $X \in \Gamma(AG)$ (see (5.83)). Thus, from (1.53), (3.22) and (5.79), we conclude that

$$\overleftarrow{d_*^{X_0}X} = \overleftarrow{[\![X,\mathcal{C}]\!]^{\phi_0}},$$

that is, the differential d_* of the Lie algebroid $(A^*G, [\![\, , \,]\!]_*, \rho_*)$ is given by (3.70) and

$$[\![\, , \,]\!]_* = [\![\, , \,]\!]_{*\mathcal{C}}, \qquad \rho_* = \rho_{*\mathcal{C}}.$$

In other words, the Jacobi groupoid $(G \rightrightarrows M, \Lambda, E, \sigma)$ integrates the triangular Jacobi bialgebroid $((AG, \phi_0), (A^*G, X_0))$ associated with the ϕ_0-canonical section $\mathcal{C}$.

Future directions

• We have explained the fundamental role played by Poisson brackets in Physics. Two natural ways for Poisson algebras to arise from a manifold M are through Poisson structures or presymplectic structures (closed 2-forms) on M. Both structures are examples of Dirac structures in the sense of Courant-Weinstein [15, 17]. A Dirac structure on a manifold M is a vector sub-bundle $\tilde{L}$ of the Whitney sum $TM \oplus T^*M$ which is maximally isotropic under the natural symmetric pairing on $TM \oplus T^*M$ and such that the space of sections of $\tilde{L}$, $\Gamma(\tilde{L})$, is closed under the Courant bracket $[\,,\,]\tilde{}$ on $\Gamma(TM \oplus T^*M) \cong \mathfrak{X}(M) \oplus \Omega^1(M)$. If $\tilde{L}$ is a Dirac structure on M, then $\tilde{L}$ is endowed with a Lie algebroid structure over M and the leaves of the induced Lie algebroid foliation $\mathcal{F}_{\tilde{L}}$ on M are presymplectic manifolds. In the particular case when the Dirac structure $\tilde{L}$ comes from a Poisson structure Π on M, then $\tilde{L}$ is isomorphic to the cotangent Lie algebroid associated with Π and $\mathcal{F}_{\tilde{L}}$ is just the symplectic foliation of M (for more details, see [15]).

An algebraic treatment of Dirac structures was developed by Dorfman in [26] using the notion of a complex over a Lie algebra. This treatment was applied to the study of general Hamiltonian structures and their role in integrability.

More recently, the properties of the Courant bracket $[\,,\,]\tilde{}$ have been systematized by Liu, Weinstein and Xu [77] in the definition of a Courant algebroid structure on a vector bundle $E \to M$ (see also [78, 101]). The natural example of a Courant algebroid is the Whitney sum $E = A \oplus A^*$, where the pair (A, A^*) is a Lie bialgebroid over M. On the other hand, one can introduce the notion of a Dirac structure on a Courant algebroid as an extension of the definition of a Dirac structure in the sense of Courant. Then, in [77] it is established a correspondence between Lie bialgebroids and pairs of transverse Dirac structures on Courant algebroids. Moreover, in [78] some applications to Poisson reduction and to the theory of Poisson homogeneous spaces for Poisson groupoids are given.

The correspondence between Poisson structures and symplectic groupoids plays an important role in Poisson geometry; it offers, in particular, a unifying framework for the study of hamiltonian and Poisson actions. In [6], the authors extend this correspondence to the context of Dirac structures twisted by a closed 3-form (see also [11]).

A proper definition of a Dirac structure on the vector bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$, as a version of a Dirac structure in the Jacobi setting, has been introduced by A. Wade in [113] (a $\mathcal{E}^1(M)$-Dirac structure in our terminology). A $\mathcal{E}^1(M)$-Dirac structure is a vector sub-bundle L of $\mathcal{E}^1(M)$ that is maximally isotropic under the natural symmetric pairing of $\mathcal{E}^1(M)$ and such that the space $\Gamma(L)$ is closed under a suitable bracket $[\,,\,]$ on $\Gamma(\mathcal{E}^1(M))$ (this bracket may be defined using the general algebraic constructions of Dorfman [26]). Apart from $\mathcal{E}^1(M)$-Dirac structures which come from Dirac structures on M or from Jacobi structures, other examples can be obtained from an exact Poisson structure on M, from a 1-form on M (a precontact structure in our terminology) or from a locally conformal presymplectic (l.c.p.) structure, that is, a pair (Ω, ω), where Ω is a 2-form on M, ω is a closed 1-form and $d_0\Omega = \omega \wedge \Omega$ (see [113]).

If L is a $\mathcal{E}^1(M)$-Dirac structure, $[\,,\,]_L$ is the restriction to $\Gamma(L) \times \Gamma(L)$ of the extended Courant bracket $[\,,\,]$ and ρ_L is the restriction to L of the canonical projection $\rho : \mathcal{E}^1(M) \to TM$, then the triple $(L, [\, , \,]_L, \rho_L)$ is a Lie algebroid over M (see [113]). An important remark is that the section ϕ_L of the dual bundle L^* defined by $\phi_L(e) = f$, for $e = (X, f) + (\mu, g) \in \Gamma(L)$, is a 1-cocycle of the Lie algebroid $(L, [\, , \,]_L, \rho_L)$. Therefore, we can obtain a Jacobi algebroid structure $(L, ([\, , \,]_L, \rho_L), \phi_L)$ from any $\mathcal{E}^1(M)$-Dirac structure. Anyway, since $\mathcal{E}^1(M)$-Dirac structures are closely related with Jacobi structures, it is not very surprising the presence of a Jacobi algebroid in the theory. Several aspects related with the geometry of $\mathcal{E}^1(M)$-Dirac structures were discussed by Wade in [113]. Moreover, in [50] we describe the nature of the induced structure on the leaves of the characteristic foliation of a $\mathcal{E}^1(M)$-Dirac structure.

In addition, very recently, Grabowski and Marmo [34] have introduced the notion of a Courant-Jacobi algebroid, a Jacobi version for Courant algebroids. These structures happen to be a particular case of a purely algebraic structure described in [107]. Using the results of [107], Grabowski and Marmo prove that every Jacobi bialgebroid $((A, \phi_0), (A^*, X_0))$ induces a Jacobi-Courant algebroid structure on $A \oplus A^*$ which admits a pair of transverse Dirac-Jacobi structures.

Therefore, if $(G \rightrightarrows M, \Lambda, E, \sigma)$ is a Jacobi groupoid and $((AG, \phi_0), (A^*G, X_0))$ is the corresponding Jacobi bialgebroid, a natural question arises: Is it possible to introduce the notion of a Jacobi homogeneous space of G in such a way that Dirac-Jacobi structures of the Jacobi-Courant algebroid $AG \oplus A^*G$ may be described in terms of Jacobi homogeneous spaces of G? The idea is to extend the one-to-one correspondence between Poisson homogeneous spaces of a Poisson groupoid $(G \rightrightarrows M, \Pi)$ and Dirac structures of the Courant algebroid $AG \oplus A^*G$ (see [78]). On the other hand, as in the Poisson setting, it is probable that Dirac-Jacobi structures may be applied to the Jacobi reduction.

• As we indicated in the introduction of this Thesis, in [31] and [91] (see also [103]) the authors started an investigation on the possible generalization of the concept of a Lie algebroid to affine bundles. In the terminology of

[31], the resultant structure was called a Lie affgebroid structure. Using this geometric model, in [91] the authors develop a time-dependent version of lagrangian equations on Lie algebroids. An important fact to remark in the previous construction is that a Lie affgebroid structure on an affine bundle A can be interpreted, in an equivalent way, as a Jacobi algebroid structure on the bi-dual bundle $(A^+)^*$ of A, that is, a usual Lie algebroid structure on $(A^+)^*$ and 1-cocycle (for this structure) non-vanishing at any point.

Thus, having in mind the relation between homogeneous Jacobi structures and Jacobi algebroids, it is interesting the study of affine Jacobi structures on affine bundles. In this direction, a first step has been done. In fact, in a recent work [32], we have studied affine Jacobi structures on affine bundles. More precisely, we have proved that if $\tau : A \to M$ is an affine bundle, then there exists a one-to-one correspondence between affine Jacobi structures on A and Lie algebroid structures on the affine dual A^+ of A. As a consequence, we recover the results obtained in Chapter 2 about homogeneous Jacobi structures.

Thus, the next step of this study could be to generalize the results obtained in this Memory to the affine setting. More precisely, to develop a theory of affine Lie-Jacobi groups analogous to the theory of Poisson-Lie groups. A vector space endowed with an affine Jacobi structure could be the abelian model of this new geometric object. Affine Jacobi-Lie groups should be Lie groups endowed with a Jacobi structure satisfying some compatibility conditions in such a way that Jacobi-Lie groups and vector spaces with affine Jacobi structures are examples of affine Jacobi-Lie groups. Moreover, after giving this notion, we could characterize the Lie algebras of affine Jacobi-Lie groups and obtain methods to generate non-trivial examples. A final step of this study could be to introduce the notion of an affine Jacobi groupoid (as an extension of the definition of an affine Jacobi-Lie group) and to describe the corresponding infinitesimal invariant.

• Multiplicative multivector fields on Lie groups have been thoroughly studied by Lu [80, 81]. The geometry of this structures is certainly interesting.

In particular we have that any multiplicative k-vector field Λ on a Lie group G with Lie algebra $\mathfrak{g}$ induces a k-differential, that is, a map $\delta_\Lambda : \mathfrak{g} \to \wedge^k\mathfrak{g}$ which is a 1-cocycle with respect to the adjoint representation of $\mathfrak{g}$ on $\wedge^k\mathfrak{g}$. Conversely, if the Lie group G is connected and simply connected, every k-differential can be integrated to a multiplicative k-vector field on G.

We remark that multiplicative 0-vector fields are just multiplicative functions on G and that Poisson multiplicative 2-vector fields are just Poisson-Lie group structures on G.

Therefore, it should be interesting to study the geometric properties of multiplicative multivector fields on Lie groupoids (the cases $k = 1$ and $k = 2$ have been dealt in detail in [83, 84, 85, 117, 118, 120]) and their infinitesimal invariants, the so-called k-differentials on Lie algebroids. For a k-differential on a Lie algebroid we mean a map $d : \Gamma(\wedge^* A) \to \Gamma(\wedge^{k+*-1}A)$ such that it is a derivation with respect to the wedge product on $\oplus_l \Gamma(\wedge^l A)$ as well as with respect to the Schouten bracket $[\![\, ,]\!]$. A natural generalization, based on the topics discussed in this Thesis, can be the theory of σ-multiplicative multivector fields.

Two interesting examples of 2-differentials are the following ones: i) If (A, A^*) is a Lie bialgebroid and d_* is the differential of A^* then d_* is a 2-differential of square zero; and ii) a 2-differential d on A such that $d^2 = [\![\Phi, \cdot]\!]$, where Φ is a 3-section of A such that $d\Phi = 0$. This last structure is called a quasi-Lie bialgebroid (see [100]) and it is an abstract version of twisted Poisson structures, which appear in [58] related with Poisson σ-models.

Bibliography

[1] C. Albert: Le théoréme de réduction de Marsden-Weinstein en géometrie cosymplectique et de contact, *J. Geom. Phys.*, **6** (1989), 627-649.

[2] C. Albert: Un théorème de réalisation des variétés de Jacobi, *C. R. Acad. Sci. Paris*, **317** Sér. I (1993), 77-80.

[3] K.H. Bhaskara, K.Viswanath: Poisson algebras and Poisson manifolds, *Research Notes in Mathematics*, **174**, Pitman, London, 1988.

[4] D.E. Blair: Riemannian geometry of Contact and Symplectic manifolds, *Progress in Math.*, **203**, Birkhäuser, Basel, 2002.

[5] W.M. Boothby, H.C. Wang: On contact manifolds, *Ann. of Math.*, **68** (1958), 721-734.

[6] H. Bursztyn, M. Crainic, A. Weinstein, C. Zhu: Integration of twisted Dirac brackets, *Preprint (2003), arXiv: math.DG/0303180.*

[7] F. Cantrijn, M. de León, E.A. Lacomba: Gradient vector fields on cosymplectic manifolds, *J. Phys. A: Math. Gen.*, **25** (1992), 175-188.

[8] J. Cariñena: Lie groupoids and algebroids in classical and quantum mechanics, *Symmetries in Quantum Mechanics and Quantum Optics*, University of Burgos, 1999, 67-81.

[9] J. Cariñena, E. Martínez: Lie algebroid generalization of geometric mechanics, *Lie algebroids and Related Topics in Differential Geometry*, Banach Center Publ., **54**, Warsaw, 2001, 201-215.

[10] A. S. Cattaneo, G. Felder: Poisson sigma models and symplectic groupoids, in *Quantization of singular symplectic quotients*, 61–93, *Progress in Math.*, **198**, Birkhäuser, Basel, 2001.

[11] A. S. Cattaneo, P. Xu: Integration of twisted Poisson structures, *Preprint (2003), arXiv: math.SG/0302268.*

[12] B.-Y. Chu : Symplectic homogeneus spaces, *Trans. A.M.S.*, **197** (1974), 145-159.

[13] J. Clemente-Gallardo: Applications of Lie algebroids in mechanics and control theory, *Nonlinear control in the New Millenium*, Springer, Berlin, 2000, 299-313.

[14] A. Coste, P. Dazord, A. Weinstein: Groupoïdes symplectiques, *Pub. Dép. Math. Lyon,* **2/A** (1987), 1-62.

[15] T.J. Courant: Dirac manifolds, *Trans. A.M.S.*, **319** (1990), 631-661.

[16] T.J. Courant: Tangent Lie algebroids, *J. Phys. A: Math. Gen.*, **27** (1994), 4527-4536.

[17] T.J. Courant, A. Weinstein: Beyond Poisson structures, In séminaire Sud-Rhodanien de geométrie, Travaux en Cours, **27**, Hermann, Paris, 1988, 39-49.

[18] M. Crainic, R.L. Fernandes: Integrability of Lie brackets, *Ann. of Math.*, **157** (2003), 575-620.

[19] M. Crainic, R.L. Fernandes: Integrability of Poisson brackets, *Preprint (2002), arXiv: math.DG/0210152.*

[20] M. Crainic, C. Zhu: Integration of Jacobi brackets, *Preprint (2003).*

[21] J.M. Dardié, A. Medina: Double extension symplectique d'un groupe de Lie symplectique, *Adv. Math.*, **117** (1996), 208-227.

[22] P. Dazord: Intégration d'algèbres de Lie locales et Groupoïdes de contact, *C.R. Acad. Sci. Paris*, **320** Sér. I (1995), 959-964.

[23] P. Dazord: Sur l'intégration des algèbres de Lie locales et la préquantification, *Bull. Sci. Math.*, **121** (1997), 423-462.

[24] P. Dazord, A. Lichnerowicz, Ch.M. Marle: Structure locale des variétés de Jacobi, *J. Math. Pures Appl.*, **70** (1991), 101-152.

[25] A. Diatta: Géométrie de Poisson et de contact des espaces homogènes, *Ph.D. Thesis*, University of Montpellier II, (2000).

[26] I. Dorfman: Dirac structures and integrability of nonlinear evolution equations, *Nonlinear Science: Theory and Applications*, John Wiley & Sons, Ltd., Chichester, 1993.

[27] V.G. Drinfeld: Hamiltonian Lie groups, Lie bialgebras and the geometric meaning of the classical Yang-Baxter equation, *Sov. Math. Dokl.*, **27** (1983), 68-71.

[28] V.G. Drinfeld: Quantum groups, *Proc. Internat. Congress Math., Berkeley*, (1986), vol. 1, 789-820.

[29] P. Etingof, A. Varchenko: Geometry and classification of solutions of the classical dynamical Yang-Baxter equation, *Comm. Math. Phys.*, **192** (1998), 77-120.

[30] B. Fuchssteiner: The Lie algebra structure of degenerate Hamiltonian and bi-Hamiltonian systems, *Progr. Theoret. Phys.*, **68** (1982), 1082-1104.

[31] J. Grabowski, K. Grabowska, P. Urbanski: Lie brackets on affine bundles, *Preprint (2002), arXiv: math.DG/0203112.*

[32] J. Grabowski, D. Iglesias, J.C. Marrero, E. Padrón, P. Urbanski: Affine Jacobi structures on vector and affine bundles, *Preprint (2002), arXiv: math.DG/0212052.*

[33] J. Grabowski, G. Marmo: Jacobi structures revisited, *J. Phys. A: Math. Gen.*, **34** (2001), 10975-10990.

[34] J. Grabowski, G. Marmo: The graded Jacobi algebras and (co)-homology, *J. Phys. A: Math. Gen.*, **36** (2003), 161-181.

[35] J. Grabowski, P. Urbanski: Tangent lifts of Poisson and related structures *J. Phys. A: Math. Gen.*, **28** (1995), 6743-6777.

[36] J. Grabowski, P. Urbanski: Lie algebroids and Poisson-Nijenhuis structures, *Rep. Math. Phys.*, **40** (1997), 195-208.

[37] J. Grabowski, P. Urbanski: Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids, *Ann. Global Anal. Geom.*, **15** (1997), 447-486.

[38] J. Grabowski, P. Urbanski: Algebroids-general differential calculi on vector bundles, *J. Geom. Phys.*, **31** (1999), 111-141.

[39] F. Guédira, A. Lichnerowicz: Géométrie des algébres de Lie locales de Kirillov, *J. Math. Pures Appl.*, **63** (1984), 407-484.

[40] S. Haller, T. Rybicki: On the group of diffeomorphisms preserving a locally conformal symplectic structure, *Ann. Global Anal. and Geom.*, **17** (1999), 475-502.

[41] Y. Haraguchi: Sur une généralisation des structures de contact, *Ph.D. Thesis*, Univ. du Haute Alsace, Mulhouse, (1981).

[42] P.J. Higgins, K.C.H. Mackenzie: Algebraic constructions in the category of Lie algebroids, *J. Algebra*, **129** (1990), 194-230.

[43] R. Ibáñez, M. de León, J.C. Marrero, D. Martín de Diego: Coisotropic and Legendre-Lagrangian submanifolds and conformal Jacobi morphisms, *J. Phys. A: Math. Gen.*, **30** (1997), 5427-5444.

[44] R. Ibáñez, M. de León, J.C. Marrero, E. Padrón: Nambu-Jacobi and generalized Jacobi manifolds, *J. Phys. A: Math. Gen.* **31** (1998), 1267-1286.

[45] D. Iglesias: $\mathcal{E}^1(M)$-Dirac structures and Jacobi structures, *Proceedings 8th International Conference on Differential Geometry and Its Applications. Part II*, Silesian University, Opava, 2001, 275-283.

[46] D. Iglesias, B. López, J.C. Marrero y E. Padrón: Triangular Generalized Lie Bialgebroids: Homology and cohomology theories, *Lie algebroids and Related Topics in Differential Geometry*, Banach Center Publ., **54**, Warsaw, 2001, 111-133.

[47] D. Iglesias, J.C. Marrero: Some linear Jacobi structures on vector bundles, *C. R. Acad. Sci. Paris*, **331** Sér. I (2000), 125-130.

[48] D. Iglesias, J.C. Marrero: Lie algebroids, Jacobi structures and Lie groups, *Proceedings del IX Fall Workshop on Geometry and Physics* (Vilanova i la Geltrú, 2000), Publicaciones de la RSME, **3**, 89-106.

[49] D. Iglesias, J.C. Marrero: Generalized Lie bialgebroids and Jacobi structures, *J. Geom. Phys.*, **40** (2001), 176-199.

[50] D. Iglesias, J.C. Marrero: Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures, *J. Phys. A: Math. Gen.*, **35** (2002), 4085-4104.

[51] D. Iglesias-Ponte, J.C. Marrero: Generalized Lie bialgebroids and strong Jacobi-Nijenhuis structures, *Extracta Math.*, **17** (2002), 259-271.

[52] D. Iglesias, J.C. Marrero: Generalized Lie bialgebras and Jacobi structures on Lie groups, *Israel J. Math.*, **133** (2003), 285-320.

[53] D. Iglesias-Ponte, J.C. Marrero: Jacobi groupoids and generalized Lie bialgebroids, *To appear in J. Geom. Phys. (2003)*, *arXiv: math.DG/0208032*.

[54] D. Iglesias-Ponte, J.C. Marrero: Locally conformal symplectic groupoids, *Preprint (2003)*, *arXiv: math.DG/0301108*.

[55] M.V. Karasaev: Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets, *Izvestiya Akademia Nauk SSSR Seriya Matematicheskaya*, **50** (1986) 508-538, 638, English translation: Mathematics of the USSR-Izvestiya, **28** (3) (1987), 497-527.

[56] Y. Kerbrat, Z. Souici-Benhammadi: Variétés de Jacobi et groupoïdes de contact, *C. R. Acad. Sci. Paris*, **317** Sér. I (1993), 81-86.

[57] A. Kirillov: Local Lie algebras, *Russian Math. Surveys*, **31** (1976), 55-75.

[58] C. Klimcik, T. Strobl: WZW-Poisson manifolds, *J. Geom. Phys.*, **43** (2002), 341-344.

[59] M. Kontsevich: Deformation quantization of Poisson manifolds, I, *Preprint (1997)*, *arXiv: q-alg/9709040*.

[60] Y. Kosmann-Schwarzbach: Jacobian quasi-bialgebras and quasi-Poisson Lie groups, *Contemporary Mathematics*, **132** (1992), 459-489.

[61] Y. Kosmann-Schwarzbach: Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.*, **41** (1995), 153-165.

[62] Y. Kosmann-Schwarzbach: The Lie bialgebroid of a Poisson-Nijenhuis manifold, *Lett. Math. Phys.*, **38** (1996), 421-428.

[63] Y. Kosmann-Schwarzbach, F. Magri: Poisson Lie groups and complete integrability. I. Drinfeld bialgebras, dual extensions and their canonical representations, *Ann. Inst. H. Poincaré*, **49** (1988), 433-460.

[64] Y. Kosmann-Schwarzbach, F. Magri: Poisson-Nijenhuis structures, *Ann. Inst. H. Poincaré Phys. Théor.*, **53** (1990), 35-81.

[65] J.-L. Koszul: Crochet de Schouten-Nijenhuis et cohomologie, In: *Élie Cartan et les mathématiques d'aujourd'hui*, *Astérisque*, No. Hors Sér. (1985), 257-271.

[66] M. de León, B. López, J.C. Marrero, E. Padrón: On the computation of the Lichnerowicz-Jacobi cohomology, *J. Geom. Phys.*, **44** (2003), 507-522.

[67] M. de León, J.C. Marrero, E. Padrón: On the geometric quantization of Jacobi manifolds, *J. Math. Phys.*, **38** (12) (1997), 6185-6213.

[68] L.-Ch. Li, S. Parmentier: On Dynamical Poisson Groupoids, I, *Preprint (2002), arXiv: math.DG/0209212.*

[69] P. Libermann: On symplectic and contact groupoids, *Diff. Geom. and its Applications*, Proc. Conf. Opava (Czecoslovaquia), (1992), 29-45.

[70] P. Libermann: On contact groupoids and their symplectification, *Analysis and geometry in foliated manifolds*, Santiago de Compostela, (1994), 153-176.

[71] P. Libermann: Lie algebroids and Mechanics, *Arch. Math. (Brno)*, **32** (1996), 147-162.

[72] P. Libermann, Ch. M. Marle: *Symplectic Geometry and Analytical Mechanics*, Kluwer, Dordrecht, 1987.

[73] A. Lichnerowicz: Les variétés de Poisson et leurs algébres de Lie associées, *J. Differential Geometry*, **12** (1977), 253-300.

[74] A. Lichnerowicz: Les variétés de Jacobi et leurs algébres de Lie associées, *J. Math. Pures Appl.*, **57** (1978), 453-488.

[75] A. Lichnerowicz: Représentation coadjointe quotient et espaces homogénes de contact ou localement conformément symplectiques, *J. Math. Pures Appl.*, **65** (1986), 193-224.

[76] A. Lichnerowicz, A. Medina: On Lie groups with left invariant symplectic or kählerian structures, *Lett. Math. Phys.*, **16** (1988), 225-235.

[77] Z-J. Liu, A. Weinstein, P. Xu: Manin triples for Lie bialgebroids, *J. Differential Geom.*, **45** (1997), 547-574.

[78] Z-J. Liu, A. Weinstein, P. Xu: Dirac structures and Poisson homogeneous spaces, *Comm. Math. Phys.*, **192** (1998), 121-144.

[79] Z.-J. Liu, P. Xu: Exact Lie bialgebroids and Poisson groupoids, *Geom. Funct. Anal.*, **6** (1996), 138-145.

[80] J.-H. Lu: Multiplicative and affine Poisson structures on Lie groups, *Ph.D. Thesis*, University of California at Berkeley, (1990).

[81] J.-H. Lu, A. Weinstein: Poisson Lie groups, dressing transformations and Bruhat decompositions, *J. Differential Geometry*, **31** (1990), 501-526.

[82] K.C.H. Mackenzie: *Lie groupoids and Lie algebroids in differential geometry*, Cambridge University Press, 1987.

[83] K.C.H. Mackenzie, P. Xu: Lie bialgebroids and Poisson groupoids, *Duke Math. J.*, **73** (1994), 415-452.

[84] K.C.H. Mackenzie, P. Xu: Classical lifting processes and multiplicative vector fields, *Quarterly J. Math.*, **49** (1998), 59-85.

[85] K.C.H. Mackenzie, P. Xu: Integration of Lie bialgebroids, *Topology*, **39** (2000), 445-467.

[86] S. Majid: Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, *Pacific J. of Math.*, **141** (1990), 311-332.

[87] Ch.M. Marle: Quelques propriétés des variétés de Jacobi, *Symplectic geometry and mechanics*, 125-139, Travaux en Cours, Hermann, Paris, 1985.

[88] J.C. Marrero, J. Monterde, E. Padrón: Jacobi-Nijenhuis manifolds and compatible Jacobi structures, *C. R. Acad. Sci. Paris*, **329** Sér. I (1999) 797–802.

[89] E. Martínez: Lagrangian mechanics on Lie algebroids, *Acta Appl. Math.*, **67** (2001), 295-320.

[90] E. Martínez: Hamiltonian Mechanics on Lie algebroids, *Preprint (2002)*.

[91] E. Martínez, T. Mestdag, W. Sarlet: Lie algebroid structures and Lagrangian systems on affine bundles, *J. Geom. Phys.*, **44** (2002), 70-95.

[92] A. Medina, P. Revoy: Groupes de Lie à structure symplectique invariante, in "*Symplectic Geometry, Groupoids and Integrable Systems, Séminaire Sud-Rhodanien de Géométrie*" (P. Dazord and A. Weinstein, Eds.) Math. Sci. Res. Inst. Publ., Springer-Verlag, 1991, 247-266.

[93] T. Mokri: Matched pairs of Lie algebroids, *Glasgow Math. J.*, **39** (1997), 167-181.

[94] Ngô-van-Quê: Sur l'espace de prolongement différentiable, *J. Differential Geometry*, **2** (1968), 33-40.

[95] J.M. Nunes da Costa: Compatible Jacobi manifolds: geometry and reduction, *J. Phys. A: Math. Gen.* **31** (1998), 1025-1033.

[96] J.M. Nunes da Costa: A characterization of strict Jacobi-Nijenhuis manifolds through the theory of Lie algebroids, *Rep. Math. Phys.*, **50** (2002), 339-347.

[97] J.M. Nunes da Costa, F. Petalidou: Reduction of Jacobi-Nijenhuis manifolds, *J. Geom. Phys.*, **41** (2002), 181-195.

[98] J.M. Nunes da Costa, F. Petalidou: Local structure of Jacobi-Nijenhuis manifolds, *J. Geom. Phys.*, **45** (2003), 323-367.

[99] J. Pradines: Théorie de Lie pour les groupoïdes différentiables. Calcul différentiel dans la catégorie des groupoïdes infinitésimaux, *C. R. Acad. Sci. Paris*, **264** Sér. A (1967), 245-248.

[100] D. Roytenberg: Quasi-Lie bialgebroids and twisted Poisson manifolds, *Lett. Math. Phys.*, **61** (2002), 123-137.

[101] D. Roytenberg, A. Weinstein: Courant algebroids and strongly homotopy Lie algebras, *Lett. Math. Phys.*, **46** (1998), 81-93.

[102] G. Sánchez de Alvarez: Geometric methods of Classical Mechanics applied to Control Theory, *Ph.D. Thesis*, University of California at Berkeley, (1986).

[103] W. Sarlet, T. Mestdag, E. Martínez: Lie algebroid structures on a class of affine bundles, *J. Math. Phys.*, **43** (2002), 5654-75.

[104] M.A. Semenov-Tian-Shansky: What is a classical r-matrix?, *Funct. Anal. Appl.*, **17** (1983), 259-272.

[105] M.A. Semenov-Tian-Shansky: Dressing transformations and Poisson group actions, *Publ. R.I.M.S.*, **21** (1985), 1237-1260.

[106] H.J. Sussmann: Orbits of families of vector fields and integrability of distributions, *Trans. Amer. Math. Soc.*, **180** (1973), 171-188.

[107] Y. Tan, Z.-J. Liu: Generalized Lie bialgebras, *Comm. Algebra*, **26** (1998), 2293-2319.

[108] I. Vaisman: Remarkable operators and commutation formulas on locally conformal Kähler manifolds, *Compositio Math.*, **40** (1980), 287-299.

[109] I. Vaisman: Locally conformal symplectic manifolds, *Internat. J. Math. & Math. Sci.*, **8** (3) (1985), 521-536.

[110] I. Vaisman: *Lectures on the Geometry of Poisson Manifolds*, Progress in Math., **118**, Birkhäuser, Basel, 1994.

[111] I. Vaisman: The BV-algebra of a Jacobi manifold, *Ann. Polon. Math.*, **73** (2000), 275-290.

[112] V.S. Varadarajan: *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, **102**, Springer-Verlag, New York, 1984.

[113] A. Wade: Conformal Dirac Structures, *Lett. Math. Phys.*, **53** (2000), 331-348.

[114] A. Wade: A generalization of Poisson-Nijenhuis structures, *J. Geom. Phys.*, **39** (2001) 217–232.

[115] A. Weinstein: The local structure of Poisson manifolds, *J. Differential Geometry*, **18** (1983), 523-557. Errata et Addenda **22** (1985), 255.

[116] A. Weinstein: Symplectic groupoids and Poisson manifolds, *Bulletin A.M.S.*, **16** (1987), 101-104.

[117] A. Weinstein: Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan*, **40** (1988), 705-727.

[118] A. Weinstein: Affine Poisson structures, *Internat. J. Math.*, **1** (1990), 343-360.

[119] A. Weinstein: Lagrangian Mechanics and groupoids, *Fields Inst. Comm.*, **7** (1996), 207-231.

[120] P. Xu: On Poisson groupoids, *Internat. J. Math.*, **6** (1995), 101-124.

[121] P. Xu: Gerstenhaber algebras and BV-algebras in Poisson geometry, *Comm. Math. Phys.*, **200** (1999), 545-560.

[122] S. Zakrzewski: Quantum and classical pseudogroups, I and II, *Comm. Math. Phys.*, **134** (1990), 347-370, 371-195.