SYMMETRIES OF HAMILTONIAN SYSTEMS IN (SYMPLECTIC) MECHANICS AND (k-SYMPLECTIC) FIELD THEORIES

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Table of contents

Table of contents

Introduction

Symmetries and conservation laws of regular Hamiltonian systems.

- Noether symmetries and Noether's theorem.
- Special attention: non-Noether symmetries.

First: Symplectic mechanics $\longrightarrow$ autonomous dynamical systems.

Second: k-symplectic field theories $\longrightarrow$ the simplest geom. model for describing (covariant 1st-order) classical field theories.

It is a generalization to field theories of symplectic formalism describing autonomous mechanical systems.

It gives a geometric description of field theories whose Hamiltonians depend on the fields and on the corresponding moments, but not on the space-time coordinates.

Table of contents

Symplectic Hamiltonian mechanics

Let (M, ω) be a symplectic manifold (usually $M = T^*Q$).
Let $H \in \mathrm{C}^\infty(T^*Q)$ be a *Hamiltonian function*.
(T^*Q, ω, H) is a *symplectic Hamiltonian system*.

If $\sigma \colon \mathbb{R} \to T^*Q$, and $\dot{\sigma} \colon \mathbb{R} \to T(T^*Q)$ is the canonical lifting, the *Hamilton equations* are $\quad i(\dot{\sigma})\omega = \mathrm{d}H \circ \sigma$.

In canonical coordinates (q^i, p_i) in T^*Q,

$$\frac{\partial H}{\partial q^i} = -\frac{d\sigma}{dt} \, , \; \frac{\partial H}{\partial p^i} = \frac{d\sigma}{dt} \; .$$

Proposition 2.1

Let $X \in \mathfrak{X}(T^*Q)$ and $\sigma \colon \mathbb{R} \to T^*Q$ an integral curve of X. Then σ is a solution to the Hamilton equations if, and only if, X is the (unique) solution to the equation $\quad i(X)\omega = \mathrm{d}H \quad$ (i.e., the Hamiltonian vector field, $X \equiv X_H$, associated to H).

Symmetries

Definition 2.2

- *A symmetry is a diffeomorphism $\Phi\colon T^*Q \to T^*Q$ such that, for every solution σ to the Hamilton equations, we have that $\Phi \circ \sigma$ is also a solution.*
  If $\Phi = T^*\varphi$ for $\varphi\colon Q \to Q$, then Φ is a natural symmetry.
- *An infinitesimal symmetry is a vector field $Y \in \mathfrak{X}(T^*Q)$ whose local flows are local symmetries.*
  If $Y = Z^{C*}$ for $Z \in \mathfrak{X}(Q)$, then Y is a natural inf. sym.

Proposition 2.3

- *$\Phi\colon T^*Q \to T^*Q$ is a symmetry if, and only if, $\Phi_* X_H = X_H$,*
- *$Y \in \mathfrak{X}(T^*Q)$ is an infinitesimal symmetry if, and only if, $[Y, X_H] = 0$ (or $[Y, X_H] = gX_H$, if changes of parametrization are allowed).*

Noether symmetries. Noether's theorem

Definition 2.4

- *A* Noether symmetry *is a diffeomorphism* $\Phi \colon T^*Q \to T^*Q$ *such that:*
  (i) $\Phi^*\omega = \omega$,
  (ii) $\Phi^* H = H$ *(up to a constant).*
- *An* infinitesimal Noether symmetry *is a vector field* $Y \in \mathfrak{X}(T^*Q)$ *such that:*
  (i) $\mathrm{L}(Y)\omega = 0$,
  (ii) $\mathrm{L}(Y)H = 0$.

Proposition 2.5

Every (inf.) Noether symmetry is a (inf.) symmetry.

(We consider only the infinitesimal case).

Theorem 2.6

(Noether): *Let $Y \in \mathfrak{X}(T^*Q)$ be an infinitesimal Noether symmetry. Then, for every $p \in T^*Q$, there is $U_p \ni p$, such that:*

- *The form $\alpha_{(0)} \equiv i(Y)\omega \in \Omega^1(T^*Q)$ is closed. Then there exists $f \in \mathrm{C}^\infty(U_p)$, unique up to a constant function, such that $i(Y)\omega = \mathrm{d}f$.*
- *There exists $\zeta \in \mathrm{C}^\infty(U_p)$, verifying $\mathrm{L}(Y)\theta = \mathrm{d}\zeta$, on U_p; and then $f = i(Y)\theta - \zeta$ (up to a constant function on U_p).*
- *f is a conserved quantity on U_p; that is, $\mathrm{L}(X_H)f = 0$.*

Corollary 2.7

The function f (and hence the form $\alpha_{(0)}$) is invariant by Y.

Theorem 2.8

(Inverse Noether): *For every conserved quantity f, its Hamiltonian vector field Y_f is an infinitesimal Noether symmetry.*

Higher-order Noether symmetries. Generalized Noether's theorem

W. Sarlet, F. Cantrijn, "Higher-order Noether symmetries and constants of the motion", *J. Phys. A: Math. Gen.* **14** (1981) 479-492.

Definition 2.9

$Y \in \mathfrak{X}(T^*Q)$ *is an* infinitesimal Noether symmetry of order N *if:*

1. Y *is a symmetry (that is,* $[Y, X_H] = gX_H$*).*
2. $\mathrm{L}^N(Y)\omega := \overbrace{\mathrm{L}(Y)\ldots\mathrm{L}(Y)}^{N}\omega = 0$, $\mathrm{L}^m(Y)\omega^A \neq 0$ *if* $m < N$.
3. $\mathrm{L}(Y)H = 0$.

Theorem 2.10

(Noether generalized): *Let $Y \in \mathfrak{X}(T^*Q)$ be an infinitesimal Noether symmetry of order N. For every $p \in T^*Q$, there is a neighborhood $U_p \ni p$, such that:*

1. *The form $\alpha_{(N-1)} \equiv \mathrm{L}^{N-1}(Y)\, i(Y)\omega \in \Omega^1(T^*Q)$ is closed. Then there exists $f \in \mathrm{C}^\infty(U_p)$, which is unique up to a constant function, such that $\mathrm{L}^{N-1}(Y)\, i(Y)\omega = \mathrm{d}g$.*
2. *There exists $\xi \in \mathrm{C}^\infty(U_p)$, verifying that $\mathrm{L}^N(Y)\theta = \mathrm{d}\xi$, on U_p; and then $f = i(Y)\theta - \xi$ (up to a constant function).*
3. *g is a conserved quantity on U_P; that is, $\mathrm{L}(X_H)g = 0$.*

Corollary 2.11

The function f (and hence the form $\alpha_{(N-1)}$) is invariant by Y.

Proofs

The corollary, and (1) and (2) of Thm. 2.10 follow from straightforward calculations.

For (3), we prove the case $N = 2$.

As $\mathrm{L}(Y)\, i(Y)\omega = \mathrm{d}f$ (on U_p), we have

$$
\begin{aligned}
\mathrm{L}(X_H)f &= i(X_H)\,\mathrm{L}(Y)\,i(Y)\omega \\
&= \{\mathrm{L}(Y)\,i(X_H) - i([Y,X_H])\}\,i(Y)\omega \\
&= \{-\mathrm{L}(Y)\,i(Y)\,i(X_H) + i(Y)\,i([Y,X_H])\}\omega \\
&= -\mathrm{L}(Y)\,i(Y)\mathrm{d}H + i(Y)\,i(gX_H)\omega \\
&= -\mathrm{L}(Y)\,i(Y)\mathrm{d}H + g\,i(Y)\mathrm{d}H \\
&= -\mathrm{L}^2(Y)H + g\,\mathrm{L}(Y)H = 0\ .
\end{aligned}
$$

For $N > 2$, we have to repeat the above procedure $N - 1$ times. □

Other kinds of non-Noether symmetries

Theorem 2.12

Let $Y \in \mathfrak{X}(T^*Q)$ be an infinitesimal symmetry for which $\exists N \in \mathbb{N}$ such that the following condition holds (maybe only locally):

$$\mathrm{L}^N(Y)\omega = f_0\omega + f_1\,\mathrm{L}(Y)\omega + \ldots + f_{N-1}\,\mathrm{L}^{N-1}(Y)\omega \ ,$$

where $\{f_0, \ldots, f_{N-1}\} \subset \mathrm{C}^\infty(T^*Q)$ are not all constant functions. Then the non-constant functions f_j are conserved quantities.

(Proof) We prove the simplest case $\mathrm{L}(Y)\omega = f\omega$ (f non constant).

$$\begin{aligned} \mathrm{L}(X_H)\,\mathrm{L}(Y)\omega &= \mathrm{L}([X_H, Y]\omega + \mathrm{L}(Y)\,\mathrm{L}(X_H)\omega = \mathrm{L}(gX_H)\omega = 0 \\ \mathrm{L}(X_H)(f\omega) &= (\mathrm{L}(X_H)f)\omega + f\,\mathrm{L}(X_H)\omega = (\mathrm{L}(X_H)f)\omega \ , \end{aligned}$$

therefore $\mathrm{L}(X_H)\,\mathrm{L}(Y)\omega = \mathrm{L}(X_H)(f\omega) \Longrightarrow \mathrm{L}(X_H)f = 0.$ □

Theorem 2.13

Let $Y \in \mathfrak{X}(T^*Q)$ be an infinitesimal symmetry satisfying that:

- $\mathrm{L}(Y)H = 0$.
- $\exists N \in \mathbb{N}$, *and constants* $C_0, \dots, C_{N-1}$ *(some of them non-vanishing) such that:*

$$\mathrm{L}^N(Y)\omega = C_1\,\mathrm{L}(Y)\omega + \ldots + C_{N-1}\,\mathrm{L}^{N-1}(Y)\omega\ .$$

Therefore:

1. *The form* $\beta \equiv \mathrm{L}^{N-1}(Y)\, i(Y)\omega - C_{N-1}\,\mathrm{L}^{N-2}(Y)\, i(Y)\omega - \ldots - C_1\, i(Y)\omega$ *is closed. Then, for every* $p \in (T^1_k)^*Q$, *there exist an open neighbourhood* $U_p \ni p$ *and a function* $f \in \mathrm{C}^\infty(U_p)$ *(unique up to a constant), such that* $\beta = \mathrm{d}f$.
2. f *is a conserved quantity.*

(Proof)

We prove the simplest case $\mathrm{L}^2(Y)\omega = C\,\mathrm{L}(Y)\omega$ ($C \neq 0$).

In this case $\beta = \mathrm{L}(Y)\,i(Y)\omega - C\,i(Y)\omega$ and for the item (1):

$$\mathrm{d}\beta = \mathrm{L}(Y)\mathrm{d}\,i(Y)\omega - C\mathrm{d}\,i(Y)\omega = \mathrm{L}^2(Y)\omega - C\,\mathrm{L}(Y)\omega = 0 \ .$$

For the second item:

$$\begin{aligned}
\mathrm{L}(X_H)f &= i(X_H)\{\mathrm{L}(Y)\,i(Y)\omega - C\,i(Y)\omega\} \\
&= i([Y, X_H]\,i(Y)\omega + i(Y)\,\mathrm{L}(X_H)\,i(Y)\omega - C\,i(X_H)\,i(Y)\omega \\
&= -\,i(Y)\,i([Y, X_H]\omega + i(Y)\,i([X_H, Y])\omega \\
&\quad +\mathrm{L}(Y)\,i(Y)\,\mathrm{L}(X_H)\omega + C\,i(Y)\,i(X_H)\omega \\
&= -\,i(Y)\,i(gX_H)\omega - i(Y)\,i(gX_H)\omega + C\,i(Y)\mathrm{d}H \\
&= C\,\mathrm{L}(Y)H = 0 \ .
\end{aligned}$$

□

THEOREM 2.14

Let $Y \in \mathfrak{X}(T^*Q)$ be an infinitesimal symmetry satisfying that:

$$(i) \quad \mathrm{L}(Y)H = 0 \quad ; \quad (ii) \quad \mathrm{L}(Y)\omega = C\omega \quad (C \in \mathbb{R}) \quad .$$

If $\exists\zeta \in \mathrm{C}^\infty(T^*Q)$ such that the form $\beta \equiv \zeta\, i(Y)\omega$ is closed, then:

1. *For every $p \in T^*Q$, there exist a neighbourhood $U_p \ni p$ and a function $f \in \mathrm{C}^\infty(U_p)$ (unique up to a constant), such that $\beta = \mathrm{d}f$, and f is a conserved quantity.*
2. *The function ζ verifies that $\mathrm{d}\zeta \wedge i(Y)\omega + C\zeta\omega = 0$.*

(Proof) For the first item we have

$$\begin{aligned} \mathrm{L}(X_H)f &= i(X_H)(\zeta\, i(Y)\omega) = -\zeta\, i(Y)\, i(X_H)\omega \\ &= -\zeta\, i(Y)\mathrm{d}H = -\zeta\,\mathrm{L}(Y)H = 0 \ . \end{aligned}$$

For the second one,

$$0 = \mathrm{d}[\zeta\, i(Y)\omega] = \mathrm{d}\zeta \wedge i(Y)\omega + \zeta\,\mathrm{L}(Y)\omega = \mathrm{d}\zeta \wedge i(Y)\omega + C\zeta\omega \ .$$

Theorem 2.15

If $Y \in \mathfrak{X}(T^*Q)$ is an infinitesimal symmetry and $f \equiv \mathrm{L}(Y)H \neq 0$, then f is a conserved quantity.

Proof

$$\begin{aligned}
\mathrm{L}(X_H)\,\mathrm{L}(Y)H &= \mathrm{L}(X_H)\,i(Y)\mathrm{d}H \\
&= i([X_H, Y])\mathrm{d}H + i(Y)\,\mathrm{L}(X_H)\mathrm{d}H \\
&= -g\,i(X_H)\mathrm{d}H + i(Y)\,i(X_H)\mathrm{d}^2 H + i(Y)\mathrm{d}\,i(X_H)\mathrm{d}H \\
&= -g\,i^2(X_H)\omega + i(Y)\mathrm{d}\,i^2(X_H)\omega = 0
\end{aligned}$$

□

Table of contents

k-symplectic manifolds

Definition 3.1

A k-symplectic structure *on a manifold M of dimension $N = n + kn$ is a family (ω^A, V), where each ω^A is a closed 2-form and V is an integrable nk-dimensional distribution on M such that*

$$(i) \quad \omega^A|_{V\times V} = 0, \qquad (ii) \quad \cap_{A=1}^{k} \ker \omega^A = \{0\} \ .$$

Then (M, ω^A, V) is called a k-symplectic manifold.

Theorem 3.2

(Darboux) *(M, ω^A, V) k-symplectic manifold. For every $p \in M$ there exists $U_p \ni p$, and a local chart of coordinates (q^i, p_i^A), s. t.*

$$\omega^A = dq^i \wedge dp_i^A \quad , \quad V = \left\langle \frac{\partial}{\partial p_i^1}, \ldots, \frac{\partial}{\partial p_i^k} \right\rangle_{i=1,\ldots,n} \ .$$

THE BUNDLE OF k^1-COVELOCITIES. CANONICAL STRUCTURES

Q a n-dimensional differentiable manifold.
$(T^1_k)^*Q = T^*Q \oplus \overset{k}{\dots} \oplus T^*Q \longrightarrow$ *bundle of k^1-covelocities* of Q.

$$\begin{array}{rcccrcl} \pi^A\colon (T^1_k)^*Q & \to & T^*Q & ; & \pi^1_Q\colon (T^1_k)^*Q & \to & Q \\ (q;\alpha^1_q,\dots,\alpha^k_q) & \mapsto & (q;\alpha^A_q) & ; & (q;\alpha^1_q,\dots,\alpha^k_q) & \mapsto & q \end{array}$$

(q^i), $(1\leq i\leq n)$, coordinates on $U\subset Q$, induced coordinates (q^i,p^A_i), $(1\leq A\leq k)$, on $(\pi^1_Q)^{-1}(U)$:

$$q^i(q;\alpha^1_q,\dots,\alpha^k_q) = q^i(q) \quad , \quad p^A_i(q;\alpha^1_q,\dots,\alpha^k_q) = \alpha^A_q\left(\left.\frac{\partial}{\partial q^i}\right|_q\right) \ .$$

Canonical k-symplectic structure: $((T^1_k)^*Q,\omega^A,V)$

$$\theta^A = (\pi^A)^*\theta \ , \ \omega^A = (\pi^A)^*\omega = -(\pi^A)^*\mathrm{d}\theta = -\mathrm{d}\theta^A \ ; \ V = \ker\,(\pi^1_Q)_* \ .$$

(θ, ω: Liouville 1-form and canonical symplectic form on T^*Q).

$$\theta^A = p^A_i\mathrm{d}q^i \quad , \quad \omega^A = \mathrm{d}q^i\wedge\mathrm{d}p^A_i \ .$$

For a diffeomorphism $\varphi\colon Q \to Q$, its *canonical lift* to $(T^1_k)^*Q$ is the map $(T^1_k)^*\varphi\colon (T^1_k)^*Q \;\to\; (T^1_k)^*Q$ such that

$$(T^1_k)^*\varphi(q;\alpha_{1q},\dots,\alpha_{kq}) = (\varphi^{-1}(q);\, T^*_q\varphi(\alpha_{1q}),\dots,\, T^*_q\varphi(\alpha_{kq}))$$

For $Z \in \mathfrak{X}(Q)$ with local 1-parametric group $F_s\colon Q \to Q$; the *canonical lift* of Z to $(T^1_k)^*Q$ is $Z^{C*} \in \mathfrak{X}((T^1_k)^*Q)$ generated by the local 1-parametric group $(T^1_k)^*(F_s)\colon (T^1_k)^*Q \to (T^1_k)^*Q$.

Locally, if $Z = Z^i \dfrac{\partial}{\partial q^i}$ then

$$Z^{C*} = Z^i \frac{\partial}{\partial q^i} \;-\; p^A_j \frac{\partial Z^j}{\partial q^k} \frac{\partial}{\partial p^A_k} \;.$$

The bundle of k^1-velocities. Canonical structures

$T^1_k M = TM \oplus \overset{k}{\dots} \oplus TM \longrightarrow$ *bundle of k^1-velocities* of M.

$$\begin{array}{rcccrcl} \tau^A \colon T^1_k M & \to & T^* M & ; & \tau^1_M \colon T^1_k M & \to & M \\ (q, {v_1}_q, \dots, {v_k}_q) & \mapsto & (q; v_A^q) & ; & (q, {v_1}_q, \dots, {v_k}_q) & \mapsto & q \end{array}$$

(q^i) coords. on $U \subset M$, induced coords. (q^i, p_i^A) on $(\tau^1_M)^{-1}(U)$:

$$q^i({v_1}_q, \dots, {v_k}_q) = q^i(q) \quad , \quad v^i_A({v_1}_q, \dots, {v_k}_q) = {v_A}_q(q^i)\,.$$

Definition 3.3

A k-vector field on a manifold M is a section $\mathbf{X} \colon M \longrightarrow T^1_k M$ of the projection τ^1_M.

Giving a k-vector field $\mathbf{X}$ is equivalent to giving a family of k vector fields $\mathbf{X} = (X_1, \dots, X_k)$, obtained as $X_A = \tau_A \circ \mathbf{X}$.

k-vector fields in M are associated with distributions on M.

Definition 3.4

An integral section *of the k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ passing through $x \in M$ is a map $\phi \colon U_0 \subset \mathbb{R}^k \to M$, such that*

$$\phi(0) = x \ , \ \phi_*(t)\left(\frac{\partial}{\partial t^A}\bigg|_t\right) = X_A(\phi(t)) \ ; \ \ t = (t^1, \dots, t^k) \in U_0$$

or, what is equivalent, $\mathbf{X} \circ \phi = \phi^{(1)}$ *, where $\phi^{(1)} \colon \mathbb{R}^k \to T^1_k M$ is the* first prolongation *of ϕ to $T^1_k M$ defined by*

$$\phi^{(1)}(t) = \left(\phi(0) = q, \phi_*(t)\left(\frac{\partial}{\partial t^1}\bigg|_t\right), \dots, \phi_*(t)\left(\frac{\partial}{\partial t^k}\bigg|_t\right)\right) \ .$$

$\mathbf{X}$ *is* integrable *if there is an integral section at every point of M.*

A k-vector field $\mathbf{X} = (X_1, \dots, X_k)$ is integrable $\Longleftrightarrow [X_A, X_B] = 0, \ \forall A, B$
$\Longrightarrow \{X_1, \dots, X_k\}$ define an involutive distribution on M.

k-symplectic Hamiltonian field theory

Let the k-symplectic manifold $((T^1_k)^*Q, \omega^A, V)$.
Let $H \in \mathrm{C}^\infty((T^1_k)^*Q)$ be a *Hamiltonian function*.
$((T^1_k)^*Q, \omega^A, H)$ is a *k-symplectic Hamiltonian system*.

The *Hamilton-de Donder-Weyl (HDW) equations* are

$$i(\psi^{(1)}_A)\omega^A = \mathrm{d}H \circ \psi \ . \tag{3.1}$$

for $\psi \colon \mathbb{R}^k \to (T^1_k)^*Q$. In coordinates, if $\psi(t) = (\psi^i(t), \psi^A_i(t))$,

$$\frac{\partial H}{\partial q^i} = -\frac{\partial \psi^A_i}{\partial t^A} \ , \ \frac{\partial H}{\partial p^A_i} = \frac{\partial \psi^i}{\partial t^A} \ .$$

Let $\mathfrak{X}^k_H((T^1_k)^*Q)$ be the set of k-vector fields $\mathbf{X} = (X_1, \dots, X_k)$ on $(T^1_k)^*Q$ which are solutions to the equation

$$i(X_A)\omega^A = \mathrm{d}H \ . \tag{3.2}$$

Locally, $X_A = (X_A)^i \dfrac{\partial}{\partial q^i} + (X_A)_i^B \dfrac{\partial}{\partial p_i^B}$, and (3.2) is equivalent to

$$\frac{\partial H}{\partial q^i} = -(X_A)_i^A \quad , \quad \frac{\partial H}{\partial p_i^A} = (X_A)^i \ .$$

Solutions to (3.2) always exist. They are neither unique, nor necessarily integrable. They are given by $\mathbf{X} + \ker \omega^\sharp$, where $\mathbf{X}$ is a particular solution, and

$$\begin{array}{rccc} \omega^\sharp & : & \mathfrak{X}^k((T_k^1)^*Q) & \to & \Omega^1((T_k^1)^*Q) \\ & & \mathbf{X} = (X_1, \ldots, X_k) & \mapsto & i(X_A)\omega^A \end{array} .$$

Proposition 3.5

Let $\mathbf{X} = (X_1, \ldots, X_k)$ be an integrable k-vector field in $(T_k^1)^*Q$ and $\psi \colon \mathbb{R}^k \to (T_k^1)^*Q$ an integral section of $\mathbf{X}$. Then ψ is a solution to the HDW-equations (3.1) if, and only if, $\mathbf{X} \in \mathfrak{X}^k_H((T_k^1)^*Q)$.

Conservation laws. Symmetries

N. Román-Roy, M. Salgado, S. Vilariño, "Symmetries and conservation laws in the Gunther k-symplectic formalism of field theory", *Rev. Math. Phys.* **19**(10), 1117–1147 (2007).

Definition 3.6

(Olver) *A* conservation law *of a k-symplectic Hamiltonian system is a map $\mathcal{F} = (\mathcal{F}^1, \ldots, \mathcal{F}^k)\colon (T^1_k)^*Q \to \mathbb{R}^k$ such that the divergence of $\mathcal{F} \circ \psi\colon \mathbb{R}^k \to \mathbb{R}^k$ is zero, for every $\psi\colon \mathbb{R}^k \to (T^1_k)^*Q$ solution to the HDW-equations; i.e.,* $\dfrac{\partial(\mathcal{F}^A \circ \psi)}{\partial t^A} = 0$.

Proposition 3.7

$\mathcal{F}\colon (T^1_k)^*Q \to \mathbb{R}^k$ *is a conservation law if, and only if,* $\mathrm{L}(X_A)\mathcal{F}^A = 0$, *for every integrable* $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$.

Definition 3.8

- *A* symmetry *is a diffeomorphism* $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ *such that, for every solution* ψ *to the HDW equations, we have that* $\Phi \circ \psi$ *is also a solution.*
  If $\Phi = (T^1_k)^*\varphi$ *for* $\varphi\colon Q \to Q$, *then* Φ *is a* natural symmetry.
- *An* infinitesimal symmetry *is a vector field* $Y \in \mathfrak{X}((T^1_k)^*Q)$ *whose local flows are local symmetries.*
  If $Y = Z^{C*}$ *for* $Z \in \mathfrak{X}(Q)$, *then* Y *is a* natural inf. sym.

Proposition 3.9

For every integrable $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$,

- *A diffeom.* $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ *is a symmetry if, and only if,* $\Phi_*\mathbf{X} = (\Phi_*X_1, \dots, \Phi_*X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$, *it is integrable, and its integral sections are* $\psi \circ \Phi$, *for any integral section* ψ *of* $\mathbf{X}$.
- $Y \in \mathfrak{X}((T^1_k)^*Q)$ *is an infinitesimal symmetry if, and only if,* $[Y, \mathbf{X}] = ([Y, X_1], \dots, [Y, X_k]) \in \mathfrak{X}^k_H((T^1_k)^*Q)$, *or* $[Y, \mathbf{X}] \in \ker \omega^\sharp$.

Noether symmetries. Noether's theorem

Definition 3.10

- *A* Noether symmetry *is a diffeomorphism* $\Phi\colon (T^1_k)^*Q \to (T^1_k)^*Q$ *such that:*
  (i) $\Phi^*\omega^A = \omega^A$,
  (ii) $\Phi^*H = H$ *(up to a constant).*
- *An* infinitesimal Noether symmetry *is a vector field* $Y \in \mathfrak{X}((T^1_k)^*Q)$ *such that:*
  (i) $\mathrm{L}(Y)\omega^A = 0$,
  (ii) $\mathrm{L}(Y)H = 0$.

Proposition 3.11

Every (inf.) Noether symmetry is a (inf.) symmetry.

(We consider only the infinitesimal case).

Theorem 3.12

(Noether): *Let $Y \in \mathfrak{X}((T^1_k)^*Q)$ be an infinitesimal Noether symmetry. Then, for every $p \in (T^1_k)^*Q$, there is a neighbourhood $U_p \ni p$, such that:*

- *The forms $\alpha^A_{(0)} \equiv i(Y)\omega^A \in \Omega^1((T^1_k)^*Q)$ are closed. Then there exist $f^A \in \mathrm{C}^\infty(U_p)$, unique up to constant functions, such that $i(Y)\omega^A = \mathrm{d}f^A$.*
- *There exist $\zeta^A \in \mathrm{C}^\infty(U_p)$, verifying $\mathrm{L}(Y)\theta^A = \mathrm{d}\zeta^A$, on U_p; and then $f^A = i(Y)\theta^A - \zeta^A$ (up to constant functions on U_p).*
- *$f = (f^1, \dots, f^k)$ define a conservation law on U_p; that is, for every integrable k-vector field $\mathbf{X} \in \mathfrak{X}^k_H((T^1_k)^*Q)$, we have $\mathrm{L}(X_A)f^A = 0$.*

Corollary 3.13

The functions f^A (and hence the forms $\alpha^A_{(0)}$) are invariant by Y.

Higher-order Noether symmetries. Generalized Noether's theorem

N. Román-Roy, M. Salgado, S. Vilariño, "Some kinds of non- Noether symmetries in k-symplectic field theories", (2009). (In preparation).

Definition 3.14

$Y \in \mathfrak{X}((T_k^1)^*Q)$ *is an* infinitesimal Noether symmetry of order N *if:*

1. *Y is an infinitesimal symmetry.*
2. $\mathrm{L}^N(Y)\omega^A = 0$, $\mathrm{L}^m(Y)\omega^A \neq 0$ *if* $m < N$.
3. $\mathrm{L}(Y)H = 0$.

Theorem 3.15

(Noether generalized): *Let $Y \in \mathfrak{X}((T^1_k)^*Q)$ be an infinitesimal Noether symmetry of order N. For every $p \in (T^1_k)^*Q$, there is a neighborhood $U_p \ni p$, such that:*

1. *The forms $\alpha^A_{(N-1)} \equiv \mathrm{L}^{N-1}(Y)\, i(Y)\omega^A \in \Omega^1((T^1_k)^*Q)$ are closed. Then there exist $f^A \in \mathrm{C}^\infty(U_p)$, which are unique up to constant functions, such that $\mathrm{L}^{N-1}(Y)\, i(Y)\omega^A = \mathrm{d}g^A$.*
2. *There exist $\xi^A \in \mathrm{C}^\infty(U_p)$, verifying that $\mathrm{L}^N(Y)\theta^A = \mathrm{d}\xi^A$, on U_p; and then $f^A = i(Y)\theta^A - \xi^A$ (up to constant functions).*
3. *The functions $f = (f^1, \dots, f^k)$ define a conservation law; that is, for every integrable $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$, we have that $\mathrm{L}(X_A)f^A = 0$ (on U_p).*

Corollary 3.16

The functions f^A (and hence the forms $\alpha^A_{(N-1)}$) are invariant by Y.

Proofs

The corollary, and (1) and (2) of Thm. 2.10 follow from straighforward calculations.

For (3), we prove the case $N = 2$.

As $\mathrm{L}(Y)\, i(Y)\omega^A = \mathrm{d}f^A$ (on U_p), for every $A = 1, \dots, k$, we have

$$\begin{aligned}
\mathrm{L}(X_A)f^A &= i(X_A)\,\mathrm{L}(Y)\, i(Y)\omega^A \\
&= \{\mathrm{L}(Y)\, i(X_A) - i([Y, X_A])\}\, i(Y)\omega^A \\
&= \{-\mathrm{L}(Y)\, i(Y)\, i(X_A) + i(Y)\, i([Y, X_A])\}\omega^A \\
&= -\mathrm{L}(Y)\, i(Y)\mathrm{d}H = -\mathrm{L}^2(Y)H = 0 \;.
\end{aligned}$$

For $N > 2$, we have to repeat the above procedure $N - 1$ times. □

Other kinds of non-Noether symmetries

Theorem 3.17

Let $Y \in \mathfrak{X}((T^1_k)^*Q)$ be an infinitesimal symmetry satisfying that (i) $\mathrm{L}(Y)H = 0 \quad ; \quad (ii) \quad \mathrm{L}(Y)\omega^A = C^A\omega^A \quad (C^A \in \mathbb{R}) \quad .$
If $\exists \zeta \in \mathrm{C}^\infty((T^1_k)^*Q)$ such that $\beta^A \equiv \zeta\, i(Y)\omega^A$ are closed, then

1. *$\forall p \in (T^1_k)^*Q$, there exist a neighbourhood $U_p \ni p$ and functions $f^A \in \mathrm{C}^\infty(U_p)$ (unique up to constants), such that $\beta^A = \mathrm{d}f^A$, and $f = (f^1, \dots, f^k)$ define a conservation law.*
2. *The function ζ verifies that $\mathrm{d}\zeta \wedge i(Y)\omega^A + C^A\zeta\omega^A = 0$.*

(Proof) For every integrable $\mathbf{X} = (X_1, \dots, X_k) \in \mathfrak{X}^k_H((T^1_k)^*Q)$,

$$\begin{aligned} \mathrm{L}(X_A)f^A &= i(X_A)(\zeta\, i(Y)\omega^A) = -\zeta\, i(Y)\, i(X_A)\omega^A \\ &= -\zeta\, i(Y)\mathrm{d}H = -\zeta\, \mathrm{L}(Y)H = 0\ . \end{aligned}$$

Furthermore

$$0 = \mathrm{d}[\zeta\, i(Y)\omega^A] = \mathrm{d}\zeta \wedge i(Y)\omega^A + \zeta\, \mathrm{L}(Y)\omega^A = \mathrm{d}\zeta \wedge i(Y)\omega^A + C^A\zeta\omega^A. \quad \square$$

Example: (unidimensional) wave equation

$Q = \mathbb{R}, \quad k = 2. \quad$ Coords.: $(t^1 = t, t^2 = x)$.

$$\sigma \frac{\partial^2 \phi}{\partial t^2} - \tau \frac{\partial^2 \phi}{\partial x^2} = 0,$$

Hamiltonian function: $\quad H = \frac{1}{2}\left(\frac{1}{\sigma}(p^1)^2 - \frac{1}{\tau}(p^2)^2\right)$.

Canonical forms on $(T_2^1)^* Q$:

$$\omega^1 = dq \wedge dp^1 \quad , \quad \omega^2 = dq \wedge dp^2$$

The equation $i(X_1)\omega^1 + i(X_2)\omega^2 = dH$ leads to

$$\begin{aligned} X_1 &= \frac{1}{\sigma} p^1 \frac{\partial}{\partial q} + (X_1)_1 \frac{\partial}{\partial p^1} + (X_1)_2 \frac{\partial}{\partial p^2} \\ X_2 &= -\frac{1}{\tau} p^2 \frac{\partial}{\partial q} + (X_2)_1 \frac{\partial}{\partial p^1} + (X_2)_2 \frac{\partial}{\partial p^2} \end{aligned}$$

with $0 = (X_1)_1 + (X_2)_2$.

Infinitesimal non-Noether symmetry: $Y = q\dfrac{\partial}{\partial q}$.

$$i(Y)\omega^A = q\mathrm{d}p^A \quad \Longrightarrow \quad \mathrm{L}(Y)\omega^A = \mathrm{d}q \wedge \mathrm{d}p^A = \omega^A$$

$\zeta = \dfrac{1}{q}$ verifies $\mathrm{d}\zeta \wedge i(Y)\omega^A + \zeta\omega^A = 0.$

Then $\zeta\, i(Y)\omega^A = \mathrm{d}p^A$ and $f = (p^1, p^2)$.